Approximated adjusted fractional Bayes factors: A general method for testing informative hypotheses*

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Abstract

Informative hypotheses are increasingly being used in psychological sciences because they adequately capture researchers' theories and expectations. In the Bayesian framework, the evaluation of informative hypotheses often makes use of default Bayes factors such as the fractional Bayes factor. This paper approximates and adjusts the fractional Bayes factor such that it can be used to evaluate informative hypotheses in general statistical models. In the fractional Bayes factor a fraction parameter must be specified which controls the amount of information in the data used for specifying an implicit prior. The remaining fraction is used for testing the informative hypotheses. We discuss different choices of this parameter and present a scheme for setting this parameter. Furthermore, a software package is developed to compute the approximated adjusted fractional Bayes factor. Using this software package, psychological researchers can evaluate informative hypotheses by means of Bayes factors in an easy manner. Two empirical examples are used to illustrate the procedure.

Keywords: Fractional Bayes factor, Informative hypothesis, Normal approximation, Prior sensitivity.

1 Introduction

One of the objectives of psychological studies is to test hypotheses that represent scientific expectations. The main tool that is available for this purpose is null hypothesis significance testing where the goal is to falsify a null hypothesis of "no effect". On the other hand, psychologists may expect, for example, that the learning ability of children is stronger than the learning ability of adolescents which, in turn, is stronger than the learning ability of adults, or it is expected that a patient's psychological disease would decrease after the first therapy, and decrease further after subsequent therapies. These expectations cannot be formulated by the traditional null hypothesis. Instead, such expectations can be translated to so-called informative hypotheses which assume a specific structure of the model parameters (Hoijtink, 2012). An informative hypothesis consists of equality and/or inequality constraints among the parameters of interest in a statistical model. For example, three equal parameters can be represented by an equality constrained hypothesis $H_1 : \theta_1 = \theta_2 = \theta_3$, and three ordered parameters can be represented by an inequality constrained hypothesis $H_2 : \theta_1 < \theta_2 < \theta_3$. Thus class of informative hypotheses covers a much broader range of scientific expectations than the class of standard null hypotheses. In addition, by testing competing informative hypotheses directly against each other a researcher obtains a direct answer which scientific theory is most supported by the data. The interested reader is referred to `http://informative-hypotheses.sites.uu.nl/` to the tab/publiciations/applications for an overview of psychological research in which informative hypotheses were used.

Informative hypothesis testing has drawn a lot of attention both in frequentist statistics (Barlow, Bartholomew, Bremner, & Brunk, 1972; Silvapulle & Sen, 2004) and in Bayesian statistics (Hoijtink, 2012). In the frequentist framework, hypothesis testing with inequality constraints has been studied over fifty years starting with (Bartholomew, 1959). Some recent contributions can be found in van de Schoot, Hoijtink, and Deković (2010), and Klugkist, Bullens, and Postma (2012). Bayesian evaluation of informative hypotheses by means of the Bayes factor is relatively new. A decade ago, Klugkist, Laudy, and Hoijtink (2005) started using Bayes factors to evaluate inequality constrained hypotheses in ANOVA models. Follow-up research appears in Klugkist and Hoijtink (2007) for Bayesian testing of inequality and about equality constrained hypotheses, in Mulder et al. (2009) for Bayesian informative hypothesis testing in repeated measures models, in Klugkist, Laudy, and Hoijtink (2010) for Bayesian evaluation of equality and inequality constrained hypotheses in contingency tables, and in Mulder, Hoijtink, and Klugkist (2010) for Bayesian model selection of equality and inequality constrained hypotheses in the context of multivariate normal linear models. The developments on the use of Bayes factors for informative hypothesis testing are summarized in Hoijtink (2012). However, these studies are limited to assess informative hypotheses in specific models and cannot yet be applied in other models, e.g., confirmatory factor analysis or logistic regression. More recently, van de Schoot, Hoijtink, Hallquist, and Boelen (2012) enables researchers to test inequality constrained hypotheses in structural equation models, Gu, Mulder, Deković, and Hoijtink (2014) allows to evaluate inequality constrained hypothesis in general statistical models, and Böing-Messing, Van Assen, Hofman, Hoijtink, and Mulder (2017) enables researchers to test informative hypotheses on group variances. Furthermore, the usefulness of the Bayes factor for testing hypotheses in psychological research was highlighted in various studies in a special issue on the topic (Mulder & Wagenmakers, 2016). Although these studies enable hypothesis testing in a large number of statistical models using the Bayes factor, the available methods for testing hypotheses with both equality constraints and inequalities are still limited.

The incessant debate between frequentist hypothesis testing and Bayesian hypothesis testing (Wagenmakers, 2007) has highlighted an advantage of the

Bayes factor: it quantifies the relative support in the data for one hypothesis against another (Kass & Raftery, 1995). This cannot be done using classical p-values. Psychological researchers can quantify how much the data favor a hypothesis relative to another hypothesis by means of the Bayes factor. However, the popularity of the Bayes factor is limited because of two reasons: the specification of the prior can be a difficult task, especially when prior information is weak or completely unavailable, and the computation can be very intensive when the statistical model is complex. To break these barriers, Bayesian statisticians have presented several default Bayes factors based on default priors. Default priors usually do not reflect subjective prior beliefs and have distributional forms chosen such that the Bayes factor can easily be computed. Examples for default Bayes factors are JZS Bayes factor (Jeffreys, 1961; Zellner & Siow, 1980; Rouder, Speckman, Sun, Morey, & Iverson, 2009), partial Bayes factors, Bayes factor based on expected posterior priors (Pérez & Berger, 2002), intrinsic Bayes factor (Berger & Pericchi, 1996) and fractional Bayes factor (O'Hagan, 1995). The last two Bayes factors are closely related to the so called partial Bayes factor (de Santis & Spezzaferri, 1999).

In the partial Bayes factor the data is split into two parts: one part is used as a training sample to update an improper noninformative prior distribution, and the remaining part is used to compute the Bayes factor. The training sample is proper if it renders a proper updated prior. Furthermore, the training sample is called minimal if any of its subsets is not proper (Berger & Pericchi, 2004). Both the intrinsic Bayes factor and the fractional Bayes factor use the concept of the partial Bayes factor method (de Santis & Spezzaferri, 1997, 1999). The intrinsic Bayes factor is an average of the partial Bayes factors based on all possible minimal training samples. Because of the use of all possible minimal training samples, the computation of intrinsic Bayes factor can be intensive especially when the sample size and the size of the minimal training sample are large. Alternatively, the fractional Bayes factor takes a small fraction b of the likelihood of the complete data (O'Hagan, 1995). The updated proper prior in the fractional Bayes approach is then implicitly specified from a noninformative prior and a fraction of full likelihood (Gilks, 1995; Moreno, 1997; de Santis & Spezzaferri, 1999; Mulder, 2014b). In this paper, we shall refer to updated priors following from fractional Bayes methodology as fractional priors. The remaining fraction of the likelihood is then used for testing the hypotheses of interest. As will be shown in this paper, the fractional Bayes factor is computationally easy. Recently, Fouskakis, Ntzoufras, and Draper (2015) presented power expected posterior priors, which are similar to fractional priors in the sense that both of them are specified using a fraction of a likelihood function. The main difference is that the fractional prior comes from a fraction of likelihood of the observed data, whereas the power expected posterior prior follows from a fraction of the likelihood of imaginary training data coming from a prior predictive distribution.

In this paper, we focus on the fractional Bayes factor as it stands out for its convenience of evaluating informative hypotheses (Mulder, 2014b). Recently, Mulder (2014b) proposed an adjustment of the fractional Bayes factor where the fractional prior was shifted around the null value. This approach resulted in an adjusted fractional Bayes factor that converges faster to a true inequality constrained hypothesis. However, the current applications of (adjusted) fractional Bayes factors in informative hypothesis testing are still within the class of multivariate normal linear models.

This paper proposes an approximation of a fractional Bayes factor to extend its applicability for testing informative hypotheses for more general models. These models can be generalized linear (mixed) models (McCullogh & Searle, 2001) such as logistic regression models and multilevel models, and structural equation models (Kline, 2011) such as path models, confirmatory factor analysis models and latent class models. Due to large sample theory (Gelman, Carlin, Stern, & Rubin, 2004, p.101-107), the posterior distribution of the parameters in each model can be approximated by a (multivariate) normal distribution. This paper also approximates the implicit fractional prior with a (multivariate) normal distribution as a general methodology to ensure a fast computation of the (adjusted) fractional Bayesian factor. Based on these approximations we can approximate a fractional Bayes factor to evaluate informative hypotheses in general statistical models. In addition, we discuss different choices of the fraction (O'Hagan, 1995; Gu, Mulder, & Hoijtink, 2016), which is a tuning parameter in the fractional prior, and provide a guideline for choosing this fraction. Furthermore, an important issue in Bayesian hypothesis testing is the consistency of the Bayesian procedure. Previous studies have discussed the consistency of intrinsic Bayes factor (Casella, Giron, & Moreno, 2009), fractional Bayes factor (O'Hagan, 1997; de Santis & Spezzaferri, 2001), and posterior model probabilities (Moreno, Giron, & Casella, 2015). In this paper, the consistency of the approximate adjusted fractional Bayes factor will be elaborated and illustrated.

This paper is organized as follows. Section 2 introduces the informative hypothesis in general statistical models, and illustrates how the informative hypothesis is constructed based on researchers' expectation by means of two empirical examples. Thereafter, Section 3 elaborates the specification of the adjusted fractional prior and the posterior distribution using normal approximations. Based on the specified prior and posterior distributions, the approximated adjusted fractional Bayes factor is derived and a software package is presented for the evaluation of informative hypotheses in general statistical models. In Section 4 we discuss different choices of the fraction, and conduct a sensitivity study for the fractional Bayes factors with those choices. Subsequently, Section 5 revisits the two empirical examples to show how to evaluate informative hypotheses using the proposed fractional Bayes factors. This paper ends with a short conclusion.

2 Informative hypotheses in general statistical models

A statistical model is described by the likelihood function $f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})$, where $\boldsymbol{X}$ denotes the data, $\boldsymbol{\theta}$ contains the parameters that are used to specify informative hypotheses, and $\boldsymbol{\zeta}$ contains the nuisance parameters. Informative hypotheses are constructed using equality and/or inequality constraints based on the theories or expectations of researchers. The general form of the informative hypothesis is given by

$$H_i : \boldsymbol{R}_{i_0}\boldsymbol{\theta} = \boldsymbol{r}_{i_0}, \boldsymbol{R}_{i_1}\boldsymbol{\theta} > \boldsymbol{r}_{i_1}, \tag{1}$$

where $\boldsymbol{R}_{i_0}$ and $\boldsymbol{R}_{i_1}$ are the restriction matrices for equality and inequality constraints in H_i, respectively, and $\boldsymbol{r}_{i_0}$ and $\boldsymbol{r}_{i_1}$ contain constants. Note that the number of rows in $\boldsymbol{R}_{i_0}$ equals the number of equality constraints, the number of rows in $\boldsymbol{R}_{i_1}$ equals the number of inequality constraints, and the numbers of columns in $\boldsymbol{R}_{i_0}$ and $\boldsymbol{R}_{i_1}$ equal the length of $\boldsymbol{\theta}$. For example, hypothesis $H_1 : \theta_1 = 2\theta_2 = 3\theta_3 > 4\theta_4 < 5$ corresponds to

$$\begin{aligned}
\boldsymbol{R}_{1_0}\boldsymbol{\theta} &= \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \boldsymbol{r}_{1_0}, \\
\boldsymbol{R}_{1_1}\boldsymbol{\theta} &= \begin{bmatrix} 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} > \begin{bmatrix} 0 \\ -5 \end{bmatrix} = \boldsymbol{r}_{1_1}.
\end{aligned}$$

Note that a range constraint, in which the parameters of interest are constrained between two values, can be written as two inequality constraints.

For example, hypothesis $H_2 : 0 < \theta < 1$ can be expressed by $H_2 : \boldsymbol{R}_{2_1}\theta > \boldsymbol{r}_{2_1}$, where $\boldsymbol{R}_{2_1} = (1, -1)^T$ and $\boldsymbol{r}_{2_1} = (0, -1)^T$. This hypothesis can be seen as a hypothesis where it is expected that θ is approximately equal to 0.5 with maximal deviation of 0.5, i.e., $\theta \approx 0.5 \Leftrightarrow |\theta - 0.5| < 0.5$, where the maximal deviation of 0.5 should be specified subjectively by the user.

An informative hypothesis H_i can be tested against the unconstrained hypothesis

$$H_u : \boldsymbol{\theta} \text{ is unconstrained,} \tag{2}$$

against its complement

$$H_{i_c} : \text{not } H_i, \tag{3}$$

which expresses what a researchers does not expect, or against another informative hypothesis

$$H_{i'} : \boldsymbol{R}_{i'_0}\boldsymbol{\theta} = \boldsymbol{r}_{i'_0}, \boldsymbol{R}_{i'_1}\boldsymbol{\theta} > \boldsymbol{r}_{i'_1}. \tag{4}$$

It should be noted that when an informative hypothesis H_i contains at least one equality constraint, the complement of H_i is the same as the unconstrained hypothesis H_u.

Before evaluating the informative hypotheses, the parameters of interest may need to be standardized in some situations. The need of standardization depends on the statistical model and informative hypothesis under evaluation. On the one hand, the parameters have to be standardized when comparing, e.g., coefficients in regression models and factor loadings in confirmatory factor analysis. For example, testing whether the regression coefficient θ_1 is larger than θ_2 requires the standardization of θ_1 and θ_2, because a large coefficient can also result from a large scale of the corresponding predictor. On the other hand, it may not be necessary to standard the parameters $\boldsymbol{\theta}$ if they are compared to constants, and it is undesirable to standardize the parameters $\boldsymbol{\theta}$ if they represent the means. For instance, testing whether a regression coefficient is larger than 0 or testing whether the mean of group 1 is smaller than the mean of group 2 does not require standardization. If standardization is required, Gu et al. (2014) discussed two ways to do this: (1) standardize all observed and latent variables, or (2) use standardized parameters. In the situation considered by Gu et al. (2014), there was little difference between the performances of the two methods. Therefore, researchers can use either of them if necessary.

In what follows, we will use two empirical examples to illustrate how researchers' expectations can be expressed by informative hypotheses.

Table 1: Data descriptive for variables in regression model

	y_i	x_{1i}	x_{2i}	x_{3i}
mean	965.92	5.91	35.24	16.86
standard deviation	74.82	1.36	26.76	2.27

2.1 Example 1 : multiple regression

The first example concerns a multiple regression model used in Guber (1999) to investigate the relation between the educational costs of the school and the academic performance of the students. The data were collected in 50 U.S. states (available at `www.amstat.org/publications/jse/secure/v7n2/datasets.guber.cfm`). The performance of the students is measured by the average total SAT score y_i ranging from 400 to 1600. Its predictors are the average public school expenditure x_{1i}, the percentage of students taking the SAT exams x_{2i}, and the average pupil/teacher ratio x_{3i}. The descriptives of the dependent variable y_i and independent variables x_{1i}, x_{2i} and x_{3i} are shown in Table 1. The relationship between the student performance and its predictors is given in a regression model.

$$y_i = \theta_0 + \theta_1 x_{1i} + \theta_2 x_{2i} + \theta_3 x_{3i} + \epsilon_i, \tag{5}$$

where θ_0 is the intercept, θ_1, θ_2 and θ_3 are the regression coefficients, and $\epsilon_i \sim N(0, \sigma^2)$ denotes the residuals with σ^2 being their residual variance. For this regression model, the likelihood is

$$f(\boldsymbol{X}|\boldsymbol{\theta}, \boldsymbol{\zeta}) = \prod_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\{-\frac{1}{2\sigma^2}(y_i - \theta_0 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3)^2\}, \tag{6}$$

where $n = 50$ denotes the sample size, and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$ and $\boldsymbol{\zeta} = (\theta_0, \sigma^2)$.

Guber (1999) theorized that higher education expenditures results in better performance of the students in SAT exams, which implies that the coefficient θ_1 of the predictor x_{1i} is positive. In addition, in those states with a small percentage of the students taking SATs, the students are expected to do well because they have self-selected themselves into the SAT exam which is only required by universities with a high prestige. This implies that the coefficient θ_2 of the predictor x_{2i} is negative. Furthermore, although a lower pupil/teacher ratio would be associated with better performance, a school needs to spend more money on education and therefore this predictor overlaps with the expenditures. This suggests that the coefficient θ_3 of predictor

x_{3i} is zero. Consequently, we specify the informative hypothesis:

$$H_1 : \theta_1 > 0, \theta_2 < 0, \theta_3 = 0 \tag{7}$$

with $\boldsymbol{R}_{1_0} = (0, 0, 1)$, $\boldsymbol{R}_{1_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$, $r_{1_0} = 0$, and $\boldsymbol{r}_{1_1} = (0, 0)^T$ in $H_1 : \boldsymbol{R}_{1_0}\boldsymbol{\theta} = r_{1_0}, \boldsymbol{R}_{1_1}\boldsymbol{\theta} > \boldsymbol{r}_{1_1}$. Hypothesis H_1 can be tested against its complement

$$H_{1_c} : \text{not } H_1. \tag{8}$$

2.2 Example 2: repeated measures ANOVA

We reanalyze the example of the repeated measures ANOVA used in Howell (2012, p.462) based on an experiment with relaxation therapy. The experiment investigated the duration of nine patients's migraine headaches before and after relaxation training. The duration of headaches is measured by the number of hours per week. Our example uses the data for the last two weeks of the baseline where patients received no training and the last two weeks of training. Therefore, the data shown in Table 2 consists of four dependent variables, i.e., the number of hours with a headache per week for nine patients in four weeks. The random effects model for these dependent variables is (Hox, 2010, p.83):

$$y_{ij} = \mu + \eta_i + \tau_j + \epsilon_{ij}, \tag{9}$$

where y_{ij} for $i = 1, \ldots, 9$ and $j = 1, \ldots, 4$ denotes the four dependent variables, μ denotes the grand mean, $\eta_i \sim N(0, \sigma^2_\eta)$ denotes the random difference for person i which is constant for different j, τ_j denotes the fixed measurement difference for week j which is constant for different i, and $\epsilon_{ij} \sim N(0, \sigma^2_\epsilon)$ is the measurement error with respect to person i and week j. To investigate the effect of relaxation training, we specify the individual differences with a random effect and the treatment differences with a fixed effect. Thus, the mean for each measurement is

$$\theta_j = \mu + \tau_j \tag{10}$$

and $\Sigma^4_{j=1}\tau_j = 0$.

The researchers expected a reduction of the duration of headaches after relaxation training. Furthermore, it is reasonable to expect that the mean durations are equal in the first two weeks of baseline and in the last two weeks of training to ensure that other factors do not influence the duration of headaches. These expectations can be expressed by the following informative hypothesis:

$$H_2 : \theta_1 = \theta_2 > \theta_3 = \theta_4 \tag{11}$$

Table 2: Data in repeated measures ANOVA

	Baseline		Training	
Subject	week 1	week 2	week 3	week 4
1	21	22	6	6
2	20	19	4	4
3	17	15	4	5
4	25	30	12	17
5	30	27	8	6
6	19	27	7	4
7	26	16	2	5
8	17	18	1	5
9	26	24	8	9

with $\boldsymbol{R}_{2_0} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, $\boldsymbol{R}_{2_1} = [0, 1, -1, 0]$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)^T$, $\boldsymbol{r}_{2_0} = (0, 0)^T$, and $r_{2_1} = 0$ in $H_2 : \boldsymbol{R}_{2_0}\boldsymbol{\theta} = r_{2_0}, \boldsymbol{R}_{2_1}\boldsymbol{\theta} > \boldsymbol{r}_{2_1}$. We compare this hypothesis to another informative hypothesis representing that the mean number of headache hours continually declines in the four weeks:

$$H_{2'} : \theta_1 > \theta_2 > \theta_3 > \theta_4, \tag{12}$$

which only contains inequality constraints $\boldsymbol{R}_{2'_1}\boldsymbol{\theta} > \boldsymbol{r}_{2'_1}$ with $\boldsymbol{r}_{2'_1} = (0, 0, 0)^T$ and

$$\boldsymbol{R}_{2'_1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The informative hypotheses constructed in these examples can be evaluated using Bayes factors, which will be elaborated in the next section. We will revisit these examples in Section 5 to display the results of the evaluation of these informative hypotheses.

3 Approximated adjusted fractional Bayes factors

The Bayes factor is the corner-stone of Bayesian hypothesis testing. It quantifies the relative evidence in the data for one hypothesis against another. The Bayes factor of an informative hypothesis H_i against another informative hypothesis $H_{i'}$ is defined by their marginal likelihood ratio (Jeffreys, 1961; Kass & Raftery, 1995):

$$BF_{ii'} = \frac{m(\boldsymbol{X}|H_i)}{m(\boldsymbol{X}|H_{i'})}. \tag{13}$$

In Bayesian hypothesis testing, the Bayes factor has a direct interpretation as the relative evidence from the data for one hypothesis against another. If $BF_{ii'} > 1$ ($BF_{ii'} < 1$), this implies that hypothesis H_i ($H_{i'}$) receives more support from the data. Specifically, if $BF_{ii'} = 5$, then the support for H_i is 5 times larger than for $H_{i'}$. For researchers who are new to Bayes factors we recommend using the guidelines for the interpretation of Bayes factors as provided by Kass and Raftery (1995). The degree of evidence in favor of H_i can be classified as unconvincing for $1 < BF_{ii'} < 3$, positive for $BF_{ii'} > 3$, strong for $BF_{ii'} > 20$, and very strong for $BF_{ii'} > 150$. However, these rules for interpreting Bayes factors are not strict and can differ in different contexts.

The informative hypothesis H_i is nested in the unconstrained hypothesis H_u which does not contain any constraints on $\boldsymbol{\theta}$. When comparing H_i to H_u we can use the encompassing prior approach of Klugkist et al. (2005) where a prior is constructed under H_i via a truncation of the unconstrained (or encompassing) prior $\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})$ under H_u. The prior under H_i is then given by $\pi_i(\boldsymbol{\theta},\boldsymbol{\zeta}) = c_i^{-1}\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})\mathbf{1}_{\boldsymbol{\Theta}_i}(\boldsymbol{\theta})$, where $c_i = \iint_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_i}\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})d\boldsymbol{\theta}d\boldsymbol{\zeta}$ is a normalizing constraint, and $\boldsymbol{\Theta}_i = \{\boldsymbol{\theta}|\boldsymbol{R}_{i_0}\boldsymbol{\theta} = \boldsymbol{r}_{i_0}, \boldsymbol{R}_{i_1}\boldsymbol{\theta} > \boldsymbol{r}_{i_1}\}$ is the parameter space of $\boldsymbol{\theta}$ in agreement with the informative hypothesis H_i. Consequently, the Bayes factor for the informative hypothesis against the unconstrained hypothesis can be expressed as:

$$\begin{aligned}
BF_{iu} &= \frac{\iint_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_i}\pi_i(\boldsymbol{\theta},\boldsymbol{\zeta})f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})d\boldsymbol{\theta}d\boldsymbol{\zeta}}{\iint\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})d\boldsymbol{\theta}d\boldsymbol{\zeta}} \\
&= \iint_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_i}\frac{\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})\cdot c_i^{-1}}{\iint\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})d\boldsymbol{\theta}d\boldsymbol{\zeta}}d\boldsymbol{\theta}d\boldsymbol{\zeta} \\
&= c_i^{-1}\iint_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_i}\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta}|\boldsymbol{X})d\boldsymbol{\theta}d\boldsymbol{\zeta} \\
&= \frac{\iint_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_i}\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta}|\boldsymbol{X})d\boldsymbol{\theta}d\boldsymbol{\zeta}}{\iint_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_i}\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})d\boldsymbol{\theta}d\boldsymbol{\zeta}},
\end{aligned} \tag{14}$$

where $\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta}|\boldsymbol{X})$ is the posterior distribution of $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$ under H_u. For example for hypothesis $H_1: \theta_1 > 0, \theta_2 < 0, \theta_3 = 0$ in (7) with equality and inequality constraints, where we denote $\boldsymbol{\theta} = (\theta_1,\theta_2,\theta_3)^T$ and $\boldsymbol{\zeta} = (\theta_0,\sigma^2)^T$, the Bayes factor of H_1 against the unconstrained alternative in (14) comes down to

$$\begin{aligned} BF_{1u} &= \frac{\iint_{\theta_1>0,\theta_2>0} \pi_u((\theta_1,\theta_2,0)^T,\boldsymbol{\zeta}|\boldsymbol{X})d\boldsymbol{\theta}d\boldsymbol{\zeta}}{\iint_{\theta_1>0,\theta_2>0} \pi_u((\theta_1,\theta_2,0)^T,\boldsymbol{\zeta})d\boldsymbol{\theta}d\boldsymbol{\zeta}} \\ &= \frac{\int_{\theta_1>0,\theta_2>0} \pi_u(\theta_1,\theta_2|\theta_3=0,\boldsymbol{X})\pi_u(\theta_3=0|\boldsymbol{X})d\boldsymbol{\theta}}{\int_{\theta_1>0,\theta_2>0} \pi_u(\theta_1,\theta_2|\theta_3=0)\pi_u(\theta_3=0)d\boldsymbol{\theta}} \\ &= \frac{Pr(\theta_1>0,\theta_2>0|\theta_3=0,\boldsymbol{X})}{Pr(\theta_1>0,\theta_2>0|\theta_3=0)}\frac{\pi_u(\theta_3=0|\boldsymbol{X})}{\pi_u(\theta_3=0)}. \end{aligned} \tag{15}$$

Further note that for a hypothesis with only equality constraints, e.g., $H_0: \theta_1=\theta_2=\theta_3=0$, expression (14) is equal to the well-known Savage-Dickey density ratio (Dickey, 1971; Wetzels, Grasman, & Wagenmakers, 2010), i.e.,

$$BF_{0u} = \frac{\pi_u(\boldsymbol{\theta}=\mathbf{0}|\boldsymbol{X})}{\pi_u(\boldsymbol{\theta}=\mathbf{0})}. \tag{16}$$

Finally, for a hypothesis with only inequality constraints, say, $H_2: \theta_1 > \theta_2 > \theta_3 > 0$, expression (14) is equal to the ratio of posterior and prior probabilities that the inequality constraints hold under H_u, i.e.,

$$BF_{2u} = \frac{P(\theta_1>\theta_2>\theta_3>0|\boldsymbol{X})}{P(\theta_1>\theta_2>\theta_3>0)}. \tag{17}$$

Thus, in order to compute the Bayes factor the unconstrained prior and corresponding unconstrained posterior need to be determined, and subsequently the unconstrained prior and posterior need to be integrated over the constrained region under the informative hypothesis. In this section we propose a novel and general approach by using normal distributions to approximate the unconstrained posterior and the unconstrained fractional prior to compute default Bayes factors.

3.1 Fractional prior and posterior

To avoid ad hoc or subjective specification of the unconstrained prior, we consider the approach of O'Hagan (1995) which is referred to as the fractional Bayes factor. A proper default prior is automatically generated by updating a noninformative improper prior $\pi_u^N(\boldsymbol{\theta},\boldsymbol{\zeta})$ using a fraction b of the likelihood (Gilks, 1995). In the fractional Bayes factor the marginal likelihood of hypothesis H_u is defined by

$$\begin{aligned} m_b^N(\boldsymbol{X}|H_u) &= \frac{m^N(\boldsymbol{X}|H_u)}{m^N(\boldsymbol{X}^b|H_u)} = \frac{\iint \pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})^N f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta}) d\boldsymbol{\theta} d\boldsymbol{\zeta}}{\iint \pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})^N f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})^b d\boldsymbol{\theta} d\boldsymbol{\zeta}} \\ &= \iint f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})^{1-b} \frac{\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})^N f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})^b}{\iint \pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})^N f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})^b d\boldsymbol{\theta} d\boldsymbol{\zeta}} d\boldsymbol{\theta} d\boldsymbol{\zeta} \\ &= \iint f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})^{1-b} \pi_u(\boldsymbol{\theta},\boldsymbol{\zeta}|\boldsymbol{X}^b) d\boldsymbol{\theta} d\boldsymbol{\zeta}, \end{aligned} \tag{18}$$

where the proper default prior is defined by

$$\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta}|\boldsymbol{X}^b) = \frac{\pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})^N f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})^b}{\iint \pi_u(\boldsymbol{\theta},\boldsymbol{\zeta})^N f(\boldsymbol{X}|\boldsymbol{\theta},\boldsymbol{\zeta})^b d\boldsymbol{\theta} d\boldsymbol{\zeta}}. \tag{19}$$

We shall refer to (19) as the fractional prior. Note that the marginal likelihood in the fractional Bayes factor in (18) is closely related to the marginal likelihood in the partial Bayes factor where a proper default prior obtained by training a noninformative prior with a small subset of the data, called a training sample, $\mathbf{X}(\ell)$, while the remaining part of the data, say, $\mathbf{X}(-\ell)$, is used for computing the marginal likelihood. The marginal likelihood in the fractional Bayes factor also abides this idea, but then by taking a fraction b of the data, denoted by $\mathbf{X}^b$, to train a noninformative prior and then use the remaining fraction of the data, $\mathbf{X}^{1-b}$, for computing the marginal likelihood in (18). The advantage of the fractional Bayes factor is that it does not depend on the exact choice of the subset of the data because a fraction of the complete data is used (de Santis & Spezzaferri, 1999; O'Hagan, 1995).

Following similar steps as in (14) and integrating the nuisance parameters out, the fractional Bayes factor of an informative hypothesis against the unconstrained hypothesis is given by (Mulder, 2014b),

$$FBF_{iu} = \frac{\int_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_i} \pi_u(\boldsymbol{\theta}|\boldsymbol{X}) d\boldsymbol{\theta}}{\int_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_i} \pi_u(\boldsymbol{\theta}|\boldsymbol{X}^b) d\boldsymbol{\theta}}. \tag{20}$$

3.2 Normal approximations of the fractional prior and posterior distributions

Due to large sample theory (e.g., Gelman et al., 2004, p. 101), the marginal posterior in the numerator of (20) can be approximated using a normal distribution where the mean is equal to the maximum likelihood estimate and the covariance matrix is equal to the inverse of the Fisher information matrix, i.e.,

$$\pi_u(\boldsymbol{\theta}|\boldsymbol{X}) \approx N(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}}_\theta), \tag{21}$$

where $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\Sigma}}_\theta$ denote the maximum likelihood estimate and covariance matrix of $\boldsymbol{\theta}$, respectively. Note that $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\Sigma}}_\theta$ can be obtained using statistical software, such as, Mplus (Muthén & Muthén, 2010) or the R-package lavaan (Rosseel, 2012). This will be further elaborated when we come back to the empirical examples in Section 5.

The fractional prior in the denominator of (20) is also centered around the maximum likelihood estimate. However, it is based on a fraction b of the data which implies an approximated covariance matrix of $\hat{\boldsymbol{\Sigma}}_\theta/b$. Consider, for example, a normally distributed data set $x_i \sim N(\theta, \sigma^2)$ with known σ^2. The posterior of θ is given by $\pi_u(\theta|X) = N(\hat{\theta}, \hat{\sigma}_\theta^2)$ where $\hat{\theta}$ equals the sample mean $\bar{x}$ and $\hat{\sigma}_\theta^2 = \sigma^2/n$. In this setting the fractional prior of θ would be $\pi_u(\theta|X^b) = N(\hat{\theta}, \hat{\sigma}_\theta^2/b) = N(\bar{x}, \sigma^2/nb)$. For this reason we propose to approximate the fractional prior according to

$$\pi_u(\boldsymbol{\theta}|\boldsymbol{X}^b) \approx N(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}}_\theta/b). \tag{22}$$

3.3 Adjusting the prior mean

It has been suggested to center the prior distribution of $\boldsymbol{\theta}$ around the focal point of interest (e.g., Zellner and Siow (1980) and Jeffreys (1961, p.268-274) for null hypothesis testing, and Mulder (2014b) for testing informative hypothesis). Suppose, for example, we evaluate $H_1 : \theta \leq 0$ against its complement $H_2 : \theta > 0$. By constructing the priors for θ under H_1 and H_2 as a truncation of an unconstrained prior that is centered around the focal point 0, the prior distribution for θ under both hypotheses are essentially equivalent; the only difference is the sign. Furthermore, by centering the prior at 0 it is assumed that small effects are more likely a priori than large effects, which is often the case in practice. A more detailed discussion on centering prior means can be found in Mulder (2014b). In this paper, we adjust the prior in (22) as follows:

$$\pi_u^*(\boldsymbol{\theta}|\boldsymbol{X}^b) = N(\boldsymbol{\theta}^*, \hat{\boldsymbol{\Sigma}}_\theta/b), \tag{23}$$

where the adjusted prior mean is given by $\boldsymbol{\theta}^* \in \boldsymbol{\Theta}_i^* = \{\boldsymbol{\theta}|\boldsymbol{R}_{i_0}\boldsymbol{\theta} = \boldsymbol{r}_{i_0}, \boldsymbol{R}_{i_1}\boldsymbol{\theta} = \boldsymbol{r}_{i_1}\}$. For each informative hypothesis, one can define a parameter space $\boldsymbol{\Theta}_i^*$ which contains one or more $\boldsymbol{\theta}^*$. For example, $H_1 : \theta_1 > 2\theta_2 > 4$ results in $\boldsymbol{\theta}^* = (4, 2)^T$, and $H_2 : \theta_1 = \theta_2$ results in $\boldsymbol{\theta}^* \in \boldsymbol{\Theta}_i^* = \{\theta_1, \theta_2|\theta_1 = \theta_2\}$ in which $\theta_1^* = \theta_2^*$ can be any value. It should be noted that the prior mean for parameters in a range constrained hypothesis is suggested being centered in the middle of the range space (Mulder, Hoijtink, & de Leeuw, 2012), because a range constraint basically implies an approximate equality, which, in terms of a restriction for the prior mean becomes an equality. For example, the range constraint $-0.2 < \theta < 0.2$ corresponds with the approximate equality $\theta \approx 0$ with maximal deviation of 0.2. Thus, the focal point is 0, and therefore we set the prior mean to $\theta^* = 0$. Below we will deal with the choice of $\boldsymbol{\theta}^*$.

The prior distribution proposed in (23) depends on the informative hypothesis under evaluation, because the prior mean $\boldsymbol{\theta}^*$ is located on the boundary of the constrained region of the informative hypothesis. When two or more informative hypotheses are under comparison, the intersection of their constrained regions must be nonempty so that a common unconstrained prior mean $\boldsymbol{\theta}^*$ exists to evaluate all informative hypotheses against the unconstrained hypothesis. A set of informative hypotheses H_i for $i = 1, \ldots, I$ are comparable if there exists at least one solution of $\boldsymbol{\theta}$ to the set of equations (Mulder et al., 2010):

$$\begin{bmatrix} \boldsymbol{R}_{1_0} \\ \boldsymbol{R}_{1_1} \end{bmatrix} \boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{r}_{1_0} \\ \boldsymbol{r}_{1_1} \end{bmatrix}, \ldots, \begin{bmatrix} \boldsymbol{R}_{I_0} \\ \boldsymbol{R}_{I_1} \end{bmatrix} \boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{r}_{I_0} \\ \boldsymbol{r}_{I_1} \end{bmatrix}. \tag{24}$$

The solution of $\boldsymbol{\theta}$ for these equations defines the parameter space $\boldsymbol{\Theta}^*$. Examples for comparable hypotheses are $H_1 : \theta = 0$ versus $H_2 : \theta > 0$ and $H_3 : \theta_1 > \theta_2 > \theta_3$ versus $H_4 : \theta_3 > \theta_2 > \theta_1$. Hypotheses $H_5 : \theta_1 = \theta_2$ versus $H_6 : \theta_1 > \theta_2 + 1$ are not comparable because there is no solution of θ_1 and θ_2 for equations $\theta_1 = \theta_2$ and $\theta_1 = \theta_2 + 1$. It should be noted that the hypothesis $H_7 : \theta_1 > 0, \theta_2 > 0, \theta_2 > \theta_1 - 1$ cannot be properly evaluated yet because a solution does not exist for equations $\theta_1 = 0$, $\theta_2 = 0$, and $\theta_2 = \theta_1 - 1$.

Adjusting the prior mean from $\hat{\boldsymbol{\theta}}$ to $\boldsymbol{\theta}^*$ results in a slight change of the posterior for $\boldsymbol{\theta}$. In particular, the posterior mean of $\hat{\boldsymbol{\theta}}$ would be slightly shifted towards the prior mean $\boldsymbol{\theta}^*$. Large sample theory however dictates that the prior has a negligible effect on the posterior for large samples. Therefore, we leave the approximated posterior for $\boldsymbol{\theta}$, given by $N(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}}_\theta)$, unaltered. Note that a similar argument is used in the BIC approximation of the Bayes factor (Schwarz, 1978; Kass & Raftery, 1995).

Based on the adjusted fractional prior distribution (23) and the posterior distribution (21), the approximated adjusted fractional Bayes factor (AAFBF) for an informative hypothesis versus the unconstrained hypothesis can be defined as:

$$AAFBF_{iu} = \frac{\int_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_i} \pi_u(\boldsymbol{\theta}|\boldsymbol{X}) d\boldsymbol{\theta}}{\int_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_i} \pi_u^*(\boldsymbol{\theta}|\boldsymbol{X}^b) d\boldsymbol{\theta}}, \tag{25}$$

where the parameter space $\boldsymbol{\Theta}_i = \{\boldsymbol{\theta}|\boldsymbol{R}_{i_0}\boldsymbol{\theta} = \boldsymbol{r}_{i_0}, \boldsymbol{R}_{i_1}\boldsymbol{\theta} > \boldsymbol{r}_{i_1}\}$ is in agreement with the informative hypothesis H_i. The computation of the AAFBF will be elaborated in Section 3.4.

3.4 Bayes factor computation

This section presents the computation of the AAFBF. First of all, we need to determine the adjusted prior mean $\boldsymbol{\theta}^*$ in (23). Finding the parameter space $\boldsymbol{\Theta}_i^*$ can be difficult for complicated informative hypotheses (Mulder et al., 2012). However, if we transform the parameters of interest using $\boldsymbol{\beta}_0 = \boldsymbol{R}_{i_0}\boldsymbol{\theta} - \boldsymbol{r}_{i_0}$ and $\boldsymbol{\beta}_1 = \boldsymbol{R}_{i_1}\boldsymbol{\theta} - \boldsymbol{r}_{i_1}$, then the informative hypothesis under consideration becomes $H_i : \boldsymbol{\beta}_0 = 0, \boldsymbol{\beta}_1 > 0$ such that we can simply specify the prior mean vector equal to zero for the new parameter vector $\boldsymbol{\beta} = (\boldsymbol{\beta}_0^T, \boldsymbol{\beta}_1^T)^T$. Note that the range constrained hypothesis, e.g., $H_1 : 0 < \theta < 1$, is an exception because as elaborated earlier the prior mean for θ is centered as $\theta^* = 0.5$ which requires $\beta_{11}^* = \theta^* = 0.5$ and $\beta_{12}^* = 1 - \theta^* = 0.5$. The specification of the prior mean for range constraints is elaborated in Appendix A. This parameter transformation was also used in Mulder (2016) for hypotheses with only inequality constraints on correlations. Here we generalize it to equality and inequality constraints on parameters in general statistical models. The parameter transformation of $\boldsymbol{\theta}$ to $\boldsymbol{\beta}$ simplifies the form of the hypothesis without changing the expectation of researchers. For instance, testing whether two parameters are equal $\theta_1 = \theta_2$ is identical to testing whether their difference is 0, i.e., $\beta_0 = \theta_1 - \theta_2 = 0$. Consequently, the adjusted fractional prior distribution and posterior distribution for the new parameter $\boldsymbol{\beta}$ are given by:

$$\pi_u^*(\boldsymbol{\beta}|\boldsymbol{X}^b) = N(\mathbf{0}, \hat{\boldsymbol{\Sigma}}_\beta/b) \tag{26}$$

and

$$\pi_u(\boldsymbol{\beta}|\boldsymbol{X}) = N(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\Sigma}}_\beta), \tag{27}$$

respectively, where $\hat{\boldsymbol{\beta}} = \boldsymbol{R}\hat{\boldsymbol{\theta}} - \boldsymbol{r}$ and $\hat{\boldsymbol{\Sigma}}_\beta = \boldsymbol{R}\hat{\boldsymbol{\Sigma}}_\theta\boldsymbol{R}^T$ with $\boldsymbol{R} = (\boldsymbol{R}_{i_0}^T, \boldsymbol{R}_{i_1}^T)^T$ and $\boldsymbol{r} = (\boldsymbol{r}_{i_0}^T, \boldsymbol{r}_{i_1}^T)^T$. Specifically, $\hat{\boldsymbol{\beta}} = (\hat{\boldsymbol{\beta}}_0^T, \hat{\boldsymbol{\beta}}_1^T)^T$ where $\hat{\boldsymbol{\beta}}_0 = \boldsymbol{R}_{i_0}\hat{\boldsymbol{\theta}} - \boldsymbol{r}_{i_0}$ and $\hat{\boldsymbol{\beta}}_1 = \boldsymbol{R}_{i_1}\hat{\boldsymbol{\theta}} - \boldsymbol{r}_{i_1}$, and $\hat{\boldsymbol{\Sigma}}_\beta = \begin{bmatrix} \hat{\boldsymbol{\Sigma}}_{\beta_0} & \hat{\boldsymbol{\Sigma}}_{01} \\ \hat{\boldsymbol{\Sigma}}_{10} & \hat{\boldsymbol{\Sigma}}_{\beta_1} \end{bmatrix}$ where $\hat{\boldsymbol{\Sigma}}_{\beta_0} = \boldsymbol{R}_{i_0}\hat{\boldsymbol{\Sigma}}_\theta\boldsymbol{R}_{i_0}^T$ and $\hat{\boldsymbol{\Sigma}}_{\beta_1} = \boldsymbol{R}_{i_1}\hat{\boldsymbol{\Sigma}}_\theta\boldsymbol{R}_{i_1}^T$.

This parameter transformation from $\boldsymbol{\theta}$ to $\boldsymbol{\beta}$ simplifies the computation of the AAFBF. First, the AAFBF for an informative hypothesis with only equality constraints, i.e., $H_i : \boldsymbol{\beta}_0 = \mathbf{0}$, compared to the unconstrained hypothesis can be obtained using the Savage-Dickey density ratio (Dickey, 1971; Wagenmakers, Lodewyckx, Kuriyal, & Grasman, 2010; Mulder, 2014b):

$$AAFBF_{iu}^0 = \frac{\pi_u(\boldsymbol{\beta}_0 = \mathbf{0}|\boldsymbol{X})}{\pi_u^*(\boldsymbol{\beta}_0 = \mathbf{0}|\boldsymbol{X}^b)}, \tag{28}$$

where $\pi_u^*(\boldsymbol{\beta}_0 = \mathbf{0}|\boldsymbol{X}^b)$ and $\pi_u(\boldsymbol{\beta}_0 = \mathbf{0}|\boldsymbol{X})$ are the densities of the prior (26) and posterior (27), respectively, for $\boldsymbol{\beta}_0$ at the point $\boldsymbol{\beta}_0 = \mathbf{0}$ under H_u. Second, the AAFBF for an informative hypothesis with only inequality constraints, i.e., $H_i : \boldsymbol{\beta}_1 > \mathbf{0}$, compared to the unconstrained hypothesis is given by (Hoijtink, 2012; Mulder, 2014b):

$$AAFBF_{iu}^1 = \frac{\int_{\boldsymbol{\beta}_1 > \mathbf{0}} \pi_u(\boldsymbol{\beta}_1|\boldsymbol{X})d\boldsymbol{\beta}_1}{\int_{\boldsymbol{\beta}_1 > \mathbf{0}} \pi_u^*(\boldsymbol{\beta}_1|\boldsymbol{X}^b)d\boldsymbol{\beta}_1}, \tag{29}$$

where $\pi_u^*(\boldsymbol{\beta}_1|\boldsymbol{X}^b)$ and $\pi_u(\boldsymbol{\beta}_1|\boldsymbol{X})$ are the prior (26) and posterior (27), respectively, for $\boldsymbol{\beta}_1$. Finally, the AAFBF for an informative hypothesis with both equality and inequality constraints, i.e., $H_i : \boldsymbol{\beta}_0 = \mathbf{0}, \boldsymbol{\beta}_1 > \mathbf{0}$, compared to the unconstrained hypothesis can be obtained via:

$$AAFBF_{iu} = \frac{\pi_u(\boldsymbol{\beta}_0 = \mathbf{0}|\boldsymbol{X})}{\pi_u^*(\boldsymbol{\beta}_0 = \mathbf{0}|\boldsymbol{X}^b)} \cdot \frac{\int_{\boldsymbol{\beta}_1 > \mathbf{0}} \pi_u(\boldsymbol{\beta}_1|\boldsymbol{\beta}_0 = \mathbf{0}, \boldsymbol{X})d\boldsymbol{\beta}_1}{\int_{\boldsymbol{\beta}_1 > \mathbf{0}} \pi_u^*(\boldsymbol{\beta}_1|\boldsymbol{\beta}_0 = \mathbf{0}, \boldsymbol{X}^b)d\boldsymbol{\beta}_1}, \tag{30}$$

where $\pi_u^*(\boldsymbol{\beta}_1|\boldsymbol{\beta}_0 = \mathbf{0}, \boldsymbol{X}^b)$ and $\pi_u(\boldsymbol{\beta}_1|\boldsymbol{\beta}_0 = \mathbf{0}, \boldsymbol{X})$ are the prior and posterior distributions of $\boldsymbol{\beta}_1$ given $\boldsymbol{\beta}_{\mathbf{0}} = \mathbf{0}$, respectively. Note that $\pi_u^*(\boldsymbol{\beta}_1|\boldsymbol{\beta}_0 = \mathbf{0}, \boldsymbol{X}^b) = N(\mathbf{0}, (\hat{\boldsymbol{\Sigma}}_{\beta_1} - \hat{\boldsymbol{\Sigma}}_{10}\hat{\boldsymbol{\Sigma}}_{\beta_0}^{-1}\hat{\boldsymbol{\Sigma}}_{01})/b)$ and $\pi_u(\boldsymbol{\beta}_1|\boldsymbol{\beta}_0 = \mathbf{0}, \boldsymbol{X}) = N(\hat{\boldsymbol{\beta}}_1 - \hat{\boldsymbol{\Sigma}}_{10}\hat{\boldsymbol{\Sigma}}_{\beta_0}^{-1}\hat{\boldsymbol{\beta}}_0, \hat{\boldsymbol{\Sigma}}_{\beta_1} - \hat{\boldsymbol{\Sigma}}_{10}\hat{\boldsymbol{\Sigma}}_{\beta_0}^{-1}\hat{\boldsymbol{\Sigma}}_{01})$.

We let $c_i^0 = \pi_u^*(\boldsymbol{\beta}_0 = \mathbf{0}|\boldsymbol{X}^b)$ and $c_i^1 = \int_{\boldsymbol{\beta}_1 > \mathbf{0}} \pi_u^*(\boldsymbol{\beta}_1|\boldsymbol{X}^b)d\boldsymbol{\beta}_1$, which can be interpreted as the relative complexities of equality constrained hypothesis and inequality constrained hypothesis, respectively, compared to H_u under prior (26). Then, in general

$$c_i = \pi_u^*(\boldsymbol{\beta}_0 = \mathbf{0}|\boldsymbol{X}^b) \cdot \int_{\boldsymbol{\beta}_1 > \mathbf{0}} \pi_u^*(\boldsymbol{\beta}_1|\boldsymbol{\beta}_0 = \mathbf{0}, \boldsymbol{X}^b)d\boldsymbol{\beta}_1 \tag{31}$$

represents the relative complexity of informative hypothesis H_i (Hoijtink, 2012; Mulder, 2014a), which is a relative measure of the size of the parameter space under an informative hypothesis in comparison to the unconstrained parameter space. For example, the relative complexity of "$\theta_1 > \theta_2$, and θ_3 unconstrained" is larger than the relative complexity of "$\theta_1 > \theta_2 > \theta_3$". This can be understood from the fact that the parameter space of the latter is a subset of the parameter space of the first. Similarly, the relative complexity of "$\theta_1 = 0$, θ_2 unconstrained" is larger than the relative complexity of "$\theta_1 = 0, \theta_2 = 0$". It is interesting to note that the relative complexity c_i^0 of an equality constrained hypothesis $H_i : \boldsymbol{\beta} = \mathbf{0}$ becomes smaller when the prior variance of $\boldsymbol{\beta}$ under H_u becomes larger. The reason is that a larger variance of the unconstrained prior implies that a larger region of the unconstrained parameter space is likely a priori, which means that H_i is simpler relative to the unconstrained hypothesis. Furthermore, we let $f_i^0 = \pi_u(\boldsymbol{\beta}_0 = \mathbf{0}|\boldsymbol{X})$ and $f_i^1 = \int_{\boldsymbol{\beta}_1 > \mathbf{0}} \pi_u(\boldsymbol{\beta}_1|\boldsymbol{X})d\boldsymbol{\beta}_1$, which can be interpreted as the measures of relative fit of the equality constrained hypothesis and inequality constrained hypothesis, respectively, compared to H_u. Then,

$$f_i = \pi_u(\boldsymbol{\beta}_0 = \mathbf{0}|\boldsymbol{X}) \cdot \int_{\boldsymbol{\beta}_1 > \mathbf{0}} \pi_u(\boldsymbol{\beta}_1|\boldsymbol{\beta}_0 = \mathbf{0}, \boldsymbol{X})d\boldsymbol{\beta}_1 \tag{32}$$

expresses the relative fit of H_i (Hoijtink, 2012; Mulder, 2014a), which implies how well a hypothesis is supported by the data compared to the unconstrained hypothesis. The relative complexity and fit in the AAFBF can be estimated based on a similar procedure presented in Gu et al. (2014) which only considers inequality constraints. We generalize the method to hypotheses with inequality as well as equality constraints to cover a considerably large spectrum of informative hypotheses that can be tested.

The computation of the AAFBF is implemented in the software package `BaIn` (Bayesian evaluation of informative hypotheses) available at `http://informative-hypotheses.sites.uu.nl/software/`. A user manual for `BaIn` is given in Appendix B. The input of `BaIn` needs the maximum likelihood estimate and covariance matrix of the parameters of interest, which can be obtained using other software packages such as Mplus (Muthén & Muthén, 2010) or the free R-package lavaan (Rosseel, 2012). Executing `BaIn` renders the AAFBF for each informative hypothesis H_i under evaluation.

The Bayes factor of an informative hypothesis H_i against its complement H_{i_c} is

$$AAFBF_{ii_c} = \frac{f_i}{c_i} / \frac{1 - f_i}{1 - c_i}, \tag{33}$$

if H_i does not contain equality constraints. Otherwise $AAFBF_{ii_c} = AAFBF_{iu}$ because the marginal likelihood of the complement of a hypothesis which contains equality constraints is equal to the marginal likelihood of the unconstrained hypothesis. For the comparison of two informative hypotheses H_i and $H_{i'}$, the AAFBF for H_i against $H_{i'}$ can be obtained by

$$AAFBF_{ii'} = AAFBF_{iu} / AAFBF_{i'u}. \tag{34}$$

Running `BaIn` for H_i and $H_{i'}$ renders $AAFBF_{iu}$ and $AAFBF_{i'u}$ such that $AAFBF_{ii'}$ can be computed using (34).

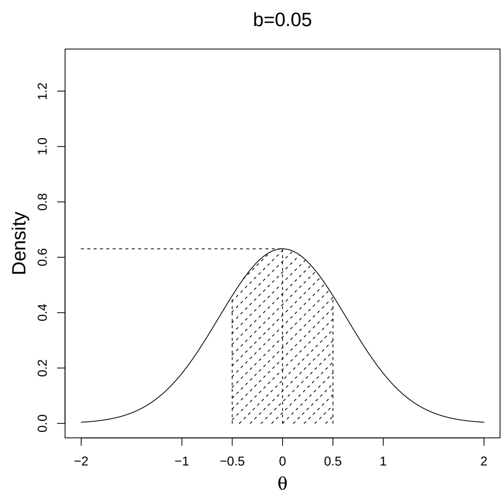
4 Choices for b

This section discusses the choices of the fraction b for the specification of fractional priors. We first show the influence of the choices of b on the AAFBF when evaluating informative hypotheses because, as with the original fractional Bayes factor (Conligliani & O'Hagan, 2000), the choice of the fraction b also plays a crucial in the AAFBF. Thereafter, two traditional choices and one novel choice of b are presented. At the end of this section, a sensitivity study is conducted to investigate the approximation error of the AAFBF relative to the actual adjusted fractional Bayes factor. It should be noted that this paper uses one common fraction b of the likelihood for prior specification. For this reason the AAFBF should only be used for testing hypotheses based on data that come from one population or balanced data with equal group sizes in the case of multiple populations, similar as the fractional Bayes factor (de Santis & Spezzaferri, 2001).

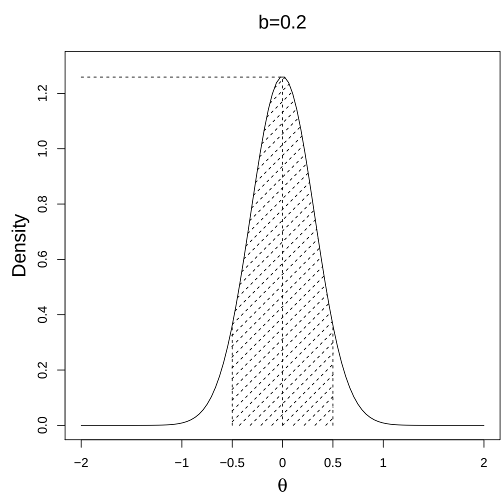
4.1 The role of b in AAFBF

The influence of fraction b on the AAFBF is different for the evaluation of equality constraints $\boldsymbol{R}_{i_0}\boldsymbol{\theta} = \boldsymbol{r}_{i_0}$ and for the evaluation of inequality constraints $\boldsymbol{R}_{i_1}\boldsymbol{\theta} > \boldsymbol{r}_{i_1}$. First of all, the fraction b is a very influential parameter when evaluating equality constraints $\boldsymbol{R}_{i_0}\boldsymbol{\theta} = \boldsymbol{r}_{i_0}$. The underlying reason is that a small (large) b implies a prior with large (small) variance such that the prior density evaluated at $\boldsymbol{R}_{i_0}\boldsymbol{\theta} = \boldsymbol{r}_{i_0}$ or $\boldsymbol{\beta}_0 = \boldsymbol{0}$ in (28) is small (large). This can be illustrated in Figure 1 in which the solid line represents the density of prior distribution $\pi_u^*(\theta|x^b) = N(0, \sigma_\theta^2/b)$ with $\sigma_\theta^2 = 0.02$ under $b = 0.05$ (a) and $b = 0.2$ (b). As can be seen, when testing hypothesis $H_1 : \theta = 0$ vs H_u, the prior density at $\theta = 0$ is 0.63 under $b = 0.05$ in Figure 1 (a), which is two times smaller than 1.26 under $b = 0.2$ in Figure 1 (b). Given an estimate of $\hat{\theta} = 0.2$ the resulting AAFBF for H_1 against H_u under $b = 0.05$ is $AAFBF_{1u} = 1.64$, whereas the AAFBF under $b = 0.2$ is $AAFBF_{1u} = 0.82$ according to equation (28).

Secondly, for range constrained hypotheses the effect of b is similar as for an equality constrained hypothesis, i.e., a small (large) b implies a large (small) AAFBF for the range constrained hypothesis against the unconstrained hypothesis. For example, the area with the shaded lines in Figure 1 represents the prior probability in line with the range constrained hypothesis $H_2 : -0.5 < \theta < 0.5$ which implies that the absolute effect is expected to be smaller than 0.5. For a small $b = 0.05$ the prior probability of $-0.5 < \theta < 0.5$ shown in Figure 1 (a) is 0.57, whereas for a large $b = 0.2$ the prior probability



(a)



(b)

Figure 1: Relative complexities under different b

in Figure 1 (b) is 0.89. Based on $\hat{\theta} = 0.2$ and equation (29) the AAFBF for H_2 against H_u under $b = 0.05$ is $AAFBF_{2u} = 1.72$, which is different from $AAFBF_{2u} = 1.11$ under $b = 0.2$.

Thirdly, the AAFBF is independent of the choice of b for inequality constrained hypotheses which do not contain range constraints. This property was proven in Mulder (2014b) and can also be seen in Figure 1 where the prior probability that the constraint of $H_3 : \theta > 0$ holds under H_u is equal to 0.5 for both choices of b.

The influence of b on AAFBF is illustrated in Figure 2 when comparing the equality constrained hypothesis $H_1 : \theta = 0$, the range constrained hypothesis $H_2 : -0.5 < \theta < 0.5$, and the inequality constrained hypothesis $H_3 : \theta > 0$, to the unconstrained hypothesis H_u. Given the estimate $\hat{\theta} = 0.2$ and variance $\hat{\sigma}^2_{\theta} = 0.02$ for θ, Figure 2 shows the AAFBF for each informative hypothesis under various $b \in (0, 0.5]$. As can be seen, the AAFBF for H_1 decreases as b increases, the AAFBF for H_2 behaves similarly as for H_1, and the AAFBF for H_3 is stable as b changes. This illustrates that the fraction b has to be carefully specified when equality constrained hypotheses and range constrained hypotheses are of interest by the researcher, while any fraction b can be used when only inequality constrained hypotheses without range constraints are formulated by the user. In what follows we will specify b in three different ways.

4.2 Traditional choices for b

Previous studies have recommended two choices for b for the fractional Bayes factor. The first one comes from Berger and Pericchi (1996) and O'Hagan (1995) who suggested using the minimal training sample for prior specification to leave maximal information in the data for hypothesis testing. This corresponds to $b = m/n$ in the fractional prior, where m is the size of the minimal training sample that makes all parameters identifiable. For example, for the one sample t test of $H_0 : \theta = 0$ where data is $x_i \sim N(\theta, \sigma^2)$, the actual adjusted fractional prior distribution for θ is $\pi^*_u(\theta|x^b) = t(0, s^2/(nb-1), nb-1)$, i.e., a Student t density with mean 0, scale parameter $s^2/(nb-1)$, and degree of freedom $nb-1$. In this case, the minimal m is 2 because $m = 1$ results in $b = 1/n$ and a degree of freedom 0, which is not allowed.

For the AAFBF we propose a similar approach to determine our first choice of b. To estimate $\boldsymbol{\beta}$ (with length J) we need at least $J+1$ observations. Therefore, our first choice of the fraction equals

$$b_{min} = (J + 1)/n, \tag{35}$$

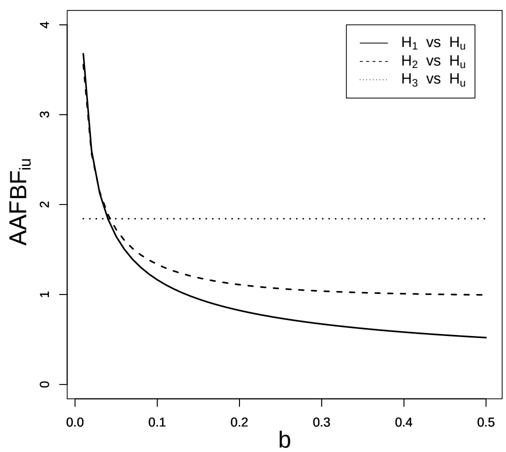


Figure 2: Influence of b on AAFBF

where J is the number of independent constraints in all the informative hypotheses under investigation, i.e., J equals the rank of $\boldsymbol{R} = (\boldsymbol{R}_{1_0}^T, \boldsymbol{R}_{1_1}^T, \ldots, \boldsymbol{R}_{I_0}^T, \boldsymbol{R}_{I_1}^T)^T$ for a set of informative hypotheses H_i for $i = 1, \ldots, I$. Thus, if $H_3 : \theta_1 = 0$ and $H_4 : \theta_1 > 0, \theta_2 > 0$ are under evaluation, for example, $J = 2$ when computing the AAFBF for each informative hypothesis against the unconstrained hypothesis because there are two independent constraints.

For the multiple regression model (5) in Section 2, $J = 3$ because $H_1 : \theta_1 > 0, \theta_2 < 0, \theta_3 = 0$ can be formulated using a vector $\boldsymbol{\beta}$ of length 3. With the sample size of $n = 50$, the first choice of the fraction b can be set as $b_{min} = 2/25$. For the repeated measures model (10), $J = 3$ based on a vector $\boldsymbol{\beta}$ of length 3 in $H_2 : \theta_1 = \theta_2 > \theta_3 = \theta_4$ and $H_{2'} : \theta_1 > \theta_2 > \theta_3 > \theta_4$, and therefore $b_{min} = 1/9$ based on sample size $n = 36$.

The second way of choosing b is (O'Hagan, 1995):

$$b_{robust} = \max\{(J+1)/n, 1/\sqrt{n}\}, \tag{36}$$

which is in general larger than the first choice. O'Hagan (1995) stated that a larger b can reduce the sensitivity of the fractional Bayes factor to the distributional form of the prior. Conligliani and O'Hagan (2000) further derived a measure of the sensitivity of the fractional Bayes factor and proved that this measure is a decreasing function of the fraction b. The second choice of b can also be applied to the AAFBF defined in (25). When setting a larger fraction b, the AAFBF becomes more similar to the non-approximated adjusted fractional Bayes factor. Thus, the AAFBF is less sensitive to the prior distribution given larger b. We will elaborate more on this topic in Section 4.4. Given the sample size of $n = 50$ in the regression model in Section 2, $b_{robust} = 1/\sqrt{50}$ is specified to evaluate hypothesis H_1. In the case of the repeated measures model with sample size $n = 36$, one can set $b_{robust} = 1/6$ for the comparison of H_2 and $H_{2'}$.

4.3 A frequentist choice for b

Gu et al. (2016) recently proposes another method of specifying b by taking into account the frequentist error probabilities. In Bayesian hypothesis testing, the probability of a Bayes factor favouring H_u when H_i is true is

$$p_1 = P(BF_{iu} < 1|H_i) \tag{37}$$

which corresponds to the Type I error probability if H_i would be a traditional null hypothesis, and the probability of a Bayes factor favouring H_i when H_u is true is

$$p_2 = P(BF_{iu} > 1|H_u). \tag{38}$$

which then corresponds to the Type II error probability. Gu et al. (2016) found that these probabilities are often quite different when using traditional choices of b in the one sample t test. This may not be preferable from a frequentist point of view where the goal typically is to control the error probabilities. Here we show how to specify b to control the error probabilities under certain conditions. First, we shall use a one sample t test to illustrate the procedure for specifying b based on this method, and then apply it to the AAFBF (28) for general statistical models. In the end, a rule of choosing b is proposed.

4.3.1 One sample t test

Consider a one sample t test for which data come from $x_i \sim N(\theta, \sigma^2)$, where θ denotes the population mean and σ^2 denotes the population variance, and the hypotheses under consideration are $H_1 : \theta = 0$ against $H_u : \theta$. The AAFBF for H_1 against H_u can be derived using equation (28):

$$AAFBF_{1u} = b^{-1/2} \exp{(-\frac{1}{2}n(\bar{x}/s)^2)}, \tag{39}$$

where $\bar{x} = \sum_{i=1}^{n} x_i/n$ and $s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$. For this AAFBF the error probabilities (37) and (38) become

$$\begin{aligned} p_1 = P(AAFBF_{1u} < 1|H_1) &= P(|\frac{\bar{x}}{s}| > \sqrt{-\log b/n} \mid H_1) \\ &\approx \frac{1}{L}\sum_{l=1}^{L} I(|\frac{\bar{x}^{(1l)}}{s^{(1l)}}| > \sqrt{-\log b/n}) \end{aligned} \quad (40)$$

and

$$\begin{aligned} p_2 = P(AAFBF_{1u} > 1|H_u) &= P(|\frac{\bar{x}}{s}| < \sqrt{-\log b/n} \mid H_u) \\ &\approx \frac{1}{L}\sum_{l=1}^{L} I(|\frac{\bar{x}^{(2l)}}{s^{(2l)}}| > \sqrt{-\log b/n}), \end{aligned} \quad (41)$$

where $\bar{x}^{(1l)}$ and $s^{(1l)}$ for $l = 1, \ldots, L$ are the mean and standard deviation of data $x_i^{(1l)}$ sampled from H_1, $\bar{x}^{(2l)}$ and $s^{(2l)}$ are the mean and standard deviation of data $x_i^{(2l)}$ sampled from H_u, and $I(\cdot)$ is the indicator function which is 1 if the argument is true and 0 otherwise. When sampling data from H_u, an expected standardized effect size, denoted by β_e, needs to be specified under H_u, i.e., $H_u : \theta = \beta_e\sigma$ so that the scaled data is sampled from $y_i \sim N(\beta_e, 1)$ under H_u where $y_i = x_i/\sigma$. Note that sampling $\frac{\bar{x}^{(2l)}}{s^{(2l)}}$ based on $x_i \sim N(\theta, \sigma^2)$ where $\theta/\sigma = \beta_e$ is identical to sampling the mean $\bar{y}^{(2l)}$ based on $y_i \sim N(\beta_e, 1)$. The specification of the standardized effect size β_e will be discussed in Section 4.3.3.

In the one sample t test, $\frac{\bar{x}}{s}$ is the observed standardized effect size known as Cohen's d (Cohen, 1992). It has sampling distributions under H_1 and H_u which can be obtained using $\frac{\bar{x}^{(1l)}}{s^{(1l)}}$ and $\frac{\bar{x}^{(2l)}}{s^{(2l)}}$, respectively. Figure 3 shows the distributions of $\frac{\bar{x}}{s}$ under $H_1 : \theta = 0$ (solid line) and $H_u : \theta = \beta_e$ (dashed line) given $\sigma^2 = 1$ and $n = 20$, where $\beta_e = 0.5$ is the pre-specified standardized effect size under H_u. Note that according to Cohen (1992) $\beta_e = .2$, .5, and .8 correspond to the small, medium, and large effects, respectively. If we use $b_{min} = 2/n$ for the one sample t test, the error probabilities in (40) and (41) become $p_1 = P(|\frac{\bar{x}}{s}| > 0.34|H_1) = 0.073$ and $p_2 = P(|\frac{\bar{x}}{s}| < 0.34|H_u) = 0.241$, whereas if we specify $b_{robust} = 1/\sqrt{n}$, the error probabilities are $p_1 = P(|\frac{\bar{x}}{s}| > 0.27|H_1) = 0.122$ and $p_2 = P(|\frac{\bar{x}}{s}| < 0.27|H_u) = 0.159$. These error probabilities are marked in Figure 3 (a) for b_{min} and (b) for b_{robust}, where the dark grey area represents p_1 and the light grey area represents p_2. As can be seen, $p_1 < p_2$ under both b_{min} and b_{robust}, which means that we are more likely to incorrectly prefer H_1 when H_u is true than incorrectly prefer H_u when H_1 is true.

In order to correct for this, Gu et al. (2016) showed how to choose b such that $p_1 = p_2$ given sample size n and effect size β_e under H_u. A direct way of obtaining such a b is proposed by Morey, Wagenmakers, and Rouder (2016) and illustrated in Figure 3 (c). As can be seen in Figure 3 (c), the distributions of $\frac{\bar{x}}{s}$ under $H_1 : \theta = 0$ and $H_u : \theta = \beta_e$ are symmetric on $\beta_e/2$. This implies that we can simply specify $\sqrt{-\log b/n} = \beta_e/2$ or equivalently $b = \exp(-n\beta_e^2/4)$ to attain equal error probabilities, because $p_1 = P(|\frac{\bar{x}}{s}| > \beta_e/2|H_1)$ is equal to $p_2 = P(|\frac{\bar{x}}{s}| < \beta_e/2|H_u)$. For example, given $n = 20$ and $\beta_e = 0.5$ under H_u in Figure 3 (c), the dark grey area for p_1 has the same size as the light grey area for p_2 when setting $b = \exp(-n\beta_e^2/4) = 0.287$. The error probabilities under this setting are $p_1 = p_2 = 0.139$.

4.3.2 General case

The method of choosing b based on equal error probabilities can be generalized to the AAFBF of any $H_i : \boldsymbol{\beta}_0 = 0$ against $H_u : \boldsymbol{\beta}_0 \neq \mathbf{0}$. Based on the adjusted fractional prior (26) and approximated posterior (27), the AAFBF in (28) is

$$AAFBF_{iu}^0 = b^{-1/2} \exp(-\frac{1}{2}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\Sigma}}_\beta^{-1}\hat{\boldsymbol{\beta}}^T). \quad (42)$$

It is interesting to note that $\sqrt{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\Sigma}}_\beta^{-1}\hat{\boldsymbol{\beta}}^T}$ in (42) is the test statistic in Wald test (Engle, 1984) which assumes that $\boldsymbol{\beta}$ is approximately normally distributed. The test statistic is not only the cornerstone in frequentist hypothesis testing, but it is also important in default Bayes factors. For example, the Bayes factor proposed by Rouder et al. (2009) for the t test is a function of t statistic, and the Bayes factor based on Zellner's g prior (Zellner & Siow, 1980) in regression models is a function of F statistic. The standardized effect size is often defined as a test statistic divided by $\sqrt{n}$ to offset the influence of the sample size (Cohen, 1992), because the effect size should not be affected by the sample size as it expresses the degree to which H_u differs from H_i. Thus, the observed standardized effect size in this case can be defined as

$$\hat{\beta}_e = \sqrt{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\Sigma}}_\beta^{-1}\hat{\boldsymbol{\beta}}^T/n}. \quad (43)$$

Then using the steps as in (40) and (41) for the one sample t test, the error probabilities of AAFBFs are defined as

$$p_1 = P(AAFBF_{iu}^0 < 1|H_i) = P(\hat{\beta}_e > \sqrt{-\log b/n}|H_i) \quad (44)$$

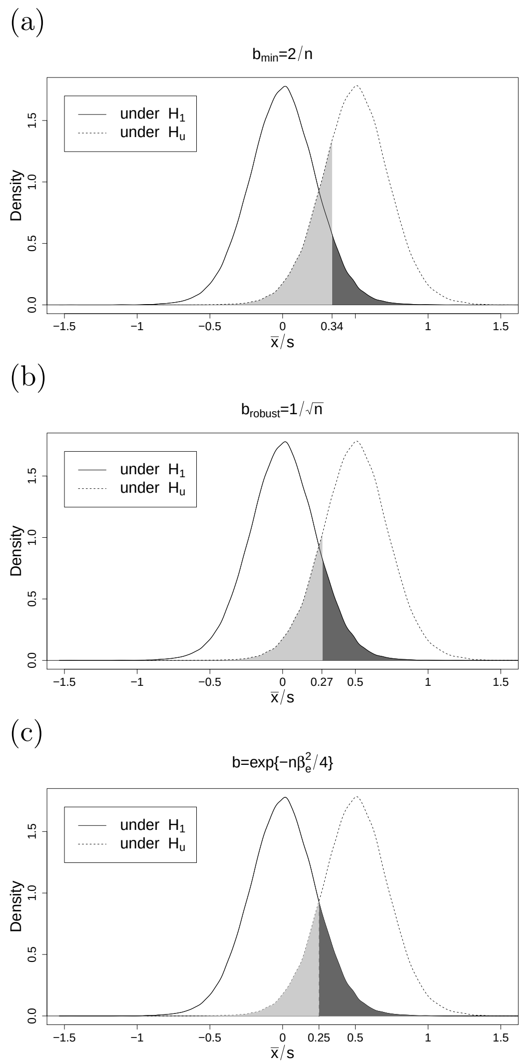


Figure 3: Sampling distributions of observed effect size $\bar{x}/s$ in one sample t test for $n = 20$ and $\beta_e = 0.5$ under H_u.

and

$$p_2 = P(AAFBF^0_{iu} > 1|H_u) = P(\hat{\beta}_e < \sqrt{-\log b/n}|H_u). \quad (45)$$

The observed standardized effect size $\hat{\beta}_e$ is usually within the interval $[0, 1]$ for equality constrained hypothesis testing, because $\hat{\beta}_e$ can be interpreted analogously as the Cohen's d or Cohen's f^2 (Cohen, 1992), which rarely exceeds 1. First, for a one sample t test $x_i \sim N(\theta, \sigma^2)$, and $H_1 : \theta = 0$ versus $H_u : \theta$, the maximum likelihood estimate of $\beta = \theta$ is $\hat{\beta} = \bar{x}$ and the standard deviation is $\hat{\sigma}_\beta = s/\sqrt{n}$. Then the observed standardized effect size (43) becomes $\hat{\beta}_e = \frac{\hat{\beta}}{\hat{\sigma}_\beta}/\sqrt{n} = \frac{\bar{x}}{s}$ which is the same as Cohen's d. Second, we consider the F test of $H_2 : \theta_1 = 0$ versus $H_u : \theta_1$ in a simple linear regression model $y_i = \theta_0 + \theta_1 x_i + \epsilon_i$, where θ_0 is the intercept, θ_1 is the regression coefficient, and $\epsilon_i \sim N(0, \sigma^2)$ is the residual. The maximum likelihood estimate of $\beta = \theta_1$ is $\hat{\beta} = r_{xy}\frac{s_y}{s_x}$ and the standard deviation is $\hat{\sigma}_\beta = \frac{\sigma}{s_x}/\sqrt{n}$, where s_x and s_y are the standard derivations of x_i and y_i, and r_{xy} is the correlation coefficient between x_i and y_i. Note that r^2_{xy} is equal to the coefficient of determination R^2 in the case of the simple linear regression model. Thus, because the coefficient of determination is equal to $R^2 = 1-\sigma^2/s^2_y$, the observed standardized effect size in (43) becomes $\hat{\beta}_e = \frac{\hat{\beta}}{\hat{\sigma}_\beta}/\sqrt{n} = r_{xy}\frac{s_y}{\sigma} = \sqrt{\frac{R^2}{1-R^2}}$, which is the square root of Cohen's $f^2 = \frac{R^2}{1-R^2}$.

Analogous to the effect size $\bar{x}/s$ in the one sample t test, the observed standardized effect size $\hat{\beta}_e$ also has sampling distributions under H_i and H_u, which are symmetric around half of the pre-specified standardized effect size β_e under H_u. Therefore, by setting $\sqrt{-\log b/n} = \beta_e/2$ or equally

$$b = \exp{(-n\beta^2_e/4)}, \quad (46)$$

the test for H_i against H_u using AAFBF has equal error probability:

$$p_1 = P(\hat{\beta}_e > \beta_e/2|H_i) = P(\hat{\beta}_e < \beta_e/2|H_u) = p_2. \quad (47)$$

How to specify β_e in (46) will be discussed in the next subsection.

4.3.3 A new rule of choosing b

Before presenting the new choice of b based on equal error probabilities, we need to deal with two issues: the range of b for consistent Bayes factors and the specification of standardized effect size β_e under H_u. The consistency of the Bayes factor is an important property in Bayesian hypothesis testing. The Bayes factor for $H_i : \boldsymbol{\beta} = 0$ against $H_u : \boldsymbol{\beta} \neq \mathbf{0}$ is consistent if it goes to infinity as sample size goes to infinity when H_i is true, and goes to 0 when

H_u is true. Morey et al. (2016) found that the prior specification based on frequentist error probabilities may result in inconsistent Bayes factors. Gu et al. (2016) showed how to resolve this by restricting the fraction b according to $b \geq 2/n$ in the one sample t test. As stated earlier in Section 4.2, $b = (J+1)/n$ is based on the minimal number of observations to specify proper priors, and therefore we will always constrain $b \geq (J+1)/n$ in the AAFBF. Furthermore, we also suggest constraining $b \leq 1/2$ because $b > 1/2$ implies that more than half of the likelihood is used for prior specification, which is undesirable in Bayesian tests (Berger & Pericchi, 1996). Consequently, the range of the fraction b is set as $b \in [(J+1)/n, 1/2]$.

To obtain the fraction b in (46) for equal error probabilities, the standardized effect size β_e under H_u has to be specified. Given any specific β_e, a fraction b in (46) can be obtained such that $p_1 = p_2$. However, in practice β_e is unknown. Therefore, a distribution for β_e is specified that covers a range of realistic effect sizes, i.e., $\beta_e \in [0,1]$ as elaborated before. Here we consider a uniform distribution $\pi^*(\beta_e) = U(0,1)$ in which every effect size from small to large is equally likely within the interval $[0,1]$ (Gu et al., 2016). Note that this choice for b would be the same as when using $\pi^*(\beta_e) = U(-1,1)$ because the choice of b is independent of the sign of the effect.

Based on the distribution of effects $\pi^*(\beta_e) = U(0,1)$, the third choice of fraction b for equal error probabilities is given by:

$$b_{freq} = E_{\pi^*(\beta_e)}[\exp(-n\beta_e^2/4)] = \int_0^1 \exp(-n\beta_e^2/4) d\beta_e. \tag{48}$$

The integration in (48) can be numerically calculated (see Gu et al., 2016). Although b_{freq} cannot always achieve equal error probabilities as we constrain $b \in [(J+1)/n, 1/2]$ and specify $\pi^*(\beta_e) = U(0,1)$, Gu et al. (2016) show that this choice results in error probabilities that are often about equal for the one sample t test. It was shown that the difference between the type I and type II error probabilities was typically smaller for this choice than when using the more traditional choices for b. We recommend the choice b_{freq} when the sample size is small, because in this case the error probabilities p_1 and p_2 are relatively large and difference between p_1 and p_2 can be quite severe. In the following subsection, we will discuss the sensitivity of AAFBF based on different choices of b.

4.4 Sensitivity to prior distributions

In Section 3, we specified the normal prior (26) for $\boldsymbol{\beta}$ in general statistical models. However, the adjusted fractional prior for the parameters in a specific model is often not normally distributed. Thus, when using a normal approximation of the fractional prior, as in the case of the AAFBF, we may misspecify the prior distribution for the parameters of interest. For example, if the parameter is a probability which is bounded in $[0, 1]$ in a binomial model, the (implicit) fractional prior would have a beta distribution. Therefore the use of the AAFBF, where the fractional prior is approximated using a normal distribution, may be different from the non-approximated adjusted fractional Bayes factor. Thus, it is useful to investigate the sensitivity of the AAFBF when the fractional prior is far from normally distributed.

O'Hagan (1995) argued that the sensitivity of the fractional Bayes factor depends on the magnitude of the fraction b. This dependence was proven by Conligliani and O'Hagan (2000). Increasing b reduces the sensitivity to the distributional form of the fractional prior. This is also the case for the adjusted fractional Bayes factor (AFBF) of Mulder (2014b), because a larger fraction b implies that more information in the data is used for prior specification, which makes the distribution of the adjusted fractional prior in the AFBF more similar to a normal distribution. This section will use two simple examples to illustrate how much difference there is between the AAFBF using the normal prior and the AFBF using the actual fractional prior. Furthermore it is shown that the AAFBF based on the different fractions show consistent behavior. In these examples, we will only focus on equality constrained hypotheses because, as elaborated earlier, the AFBF for inequality constrained hypotheses is independent of the fraction b.

The first example again concerns the one sample t test, where data come from $x_i \sim N(\theta, \sigma^2)$ with unknown mean and variance, and the hypotheses under consideration are $H_1 : \theta = 0$ against $H_u : \theta$. In the AAFBF, the default prior (26) for $\beta = \theta$ is $\pi_u^*(\beta|X^b) = N(0, s^2/nb)$, while the actual adjusted fractional prior for a normal mean has a t distribution $\pi_u^*(\beta|X^b) = t(0, s^2/(nb-1), nb-1)$ with mean of 0, variance of $s^2/(nb-1)$, and degrees of freedom of $nb-1$. It is well known that the t distribution has heavier tail than the normal distribution such that the density at the mode $\beta = 0$ from the normal distribution is larger than the density from the t distribution. Furthermore, as the fraction b increases, the degrees of freedom $nb-1$ increase such that the t distribution $t(0, s^2/(nb-1), nb-1)$ becomes more similar to the normal distribution $N(0, s^2/nb)$. This implies that for a larger b the AAFBF where the default prior has a normal distribution performs more similarly as the AFBF under the actual fractional prior. This is illustrated in Figure 4.

Figure 4 shows the logarithms of AFBFs and AAFBFs for H_1 versus H_u under different observed effect sizes $\bar{x}/s = 0, 0.1, 0.2$, and different fractions

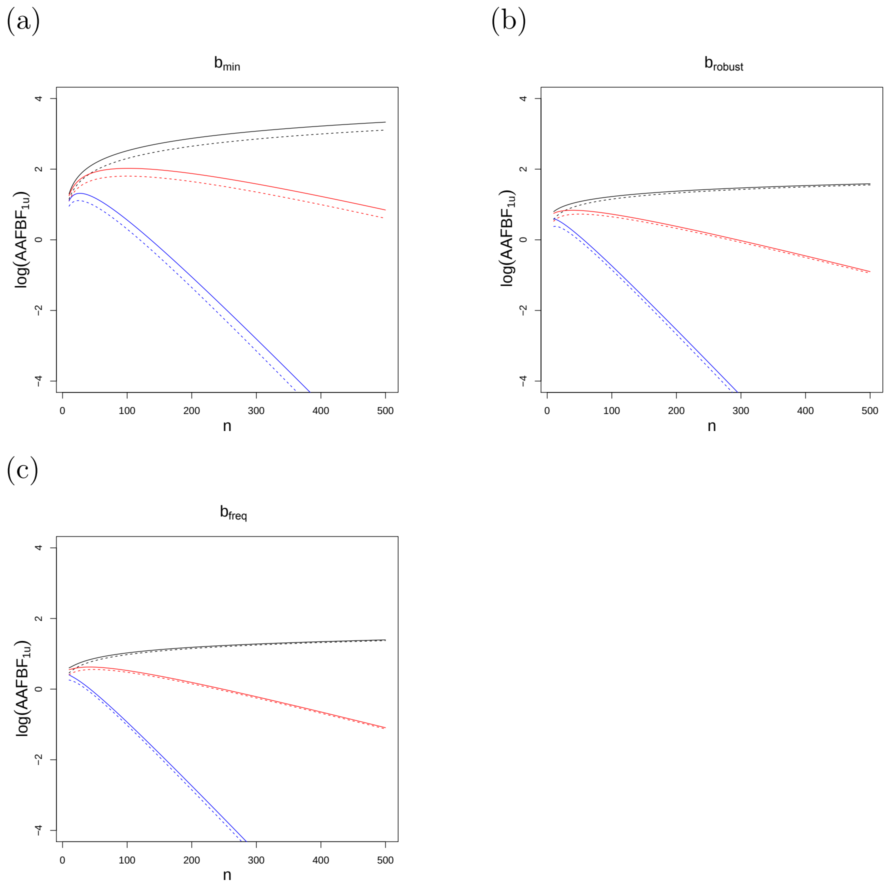


Figure 4: The logarithms of the AFBF with a Student t prior (solid line) and the AAFBF with a normal prior (dashed line). The dark, red, and green lines correspond to the logarithms of Bayes factors under observed effect sizes $\bar{x}/s = 0$, 0.1, and 0.2, respectively.

b_{min}, b_{robust}, and b_{freq}. The sample size n varies from 10 to 500. First, as can be seen in Figure 4 (a), based on b_{min} the logarithms of AAFBFs under the normal prior distribution (dashed line) differ substantially from the logarithms of AFBFs under the t prior distribution (solid line). This difference does not decrease as n increases because when setting $b_{min} = 2/n$ the degree of freedom in the t distribution is 1, which is independent of n. This suggests high sensitivity to the functional form of the prior distribution. Second, Figure 4 (b) shows that based on b_{robust} there is not much difference between the logarithms of AAFBFs and AFBFs. This implies that the choice of b_{robust} results in less sensitivity to the functional form of the prior distribution than b_{min}. Third, Figure 4 (c) demonstrates the logarithms of AAFBFs and AFBFs under b_{freq}. As can be seen, with b_{freq} there is no sensitivity either.

It is interesting to note that Figure 4 also illustrates the consistency of AAFBFs. The consistency in this example requires that as sample size goes to infinity the AAFBF for H_1 against H_u approaches to infinity when the observed effect size is equal to 0 and the AAFBF goes to zero when the observed effect size is unequal to 0. As can be seen in Figure 4, for an observed effect size $\bar{x}/s = 0$ the logarithm of the AAFBF (black lines) in each figure goes to infinity as sample size n increases. Conversely, the logarithms of the AAFBF based on an observed effect size of $\bar{x}/s = 0.1$ (red lines) and $\bar{x}/s = 0.2$ (blue lines) diverge to minus infinity, which implies decisive evidence for the true unconstrained hypothesis as the sample size goes to infinity.

Next, we consider a binomial model, where data come from $x \sim Bin(n, p)$. The hypotheses under evaluation are $H_2 : p = 0.4$ against $H_u : 0 \leq p \leq 1$. Since H_2 is nested in H_u, we can use the AAFBF (28) to evaluate H_2 against H_u. Given data $x \sim Bin(n, p)$, the estimate of $\beta = p - 0.4$ is $\hat{\beta} = x/n - 0.4$ and the variance is $\hat{\sigma}^2_\beta = \frac{x(n-x)}{n^2(n+1)}$, and therefore the normal adjusted fractional prior (26) is $\pi^*_u(\beta|X^b) = N(0, \frac{bx(n-x)}{n^2(n+1)})$. On the other hand, following the idea of adjusted fractional Bayes factors the fractional prior has a beta distribution, i.e., $p = \beta + 0.4 \sim Beta(0.4nb, 0.6nb)$ which has a mean of 0.4 and thus β has a prior mean of 0. Note that this prior is centered on the focal point of 0.4 in H_2.

Figure 5 draws the lines of the logarithms of the AFBFs and AAFBFs for H_2 against H_u as the sample size n increases from 10 to 500. The observed data are $x = 0.4n, 0.5n, 0.6n$. As can be seen in Figure 5 there is a considerable smaller approximation error of the AAFBF with respect to the AFBF in comparison to the first example in Figure 4. Again, the

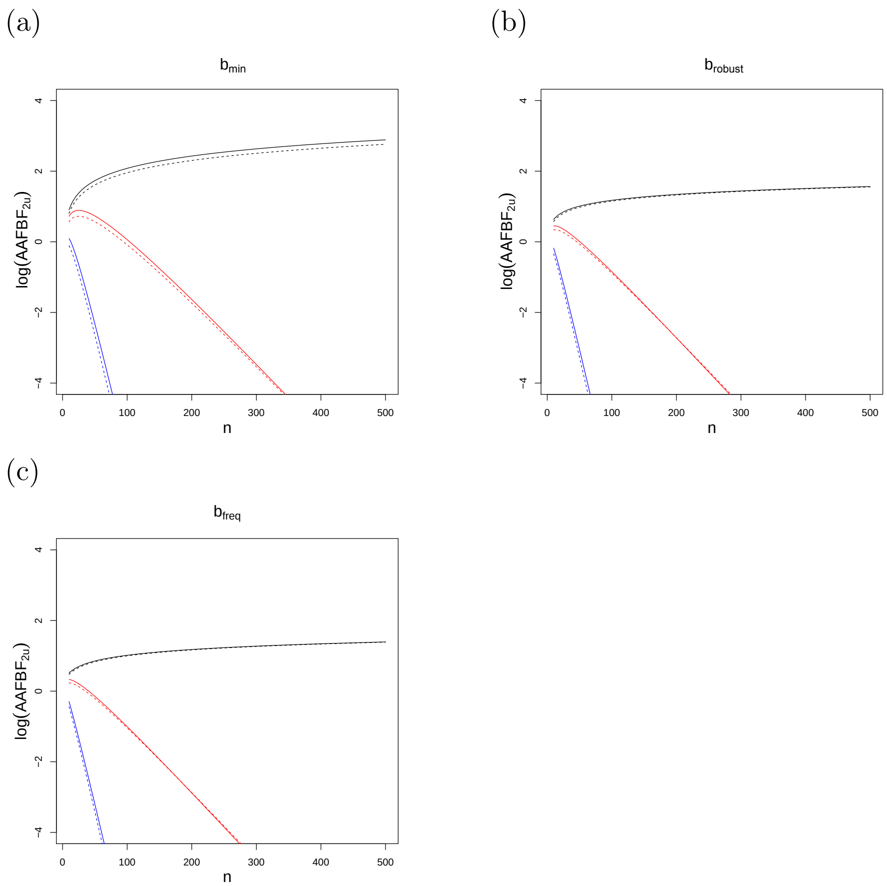


Figure 5: The logarithms of the AFBF with a Beta prior (solid line) and the AAFBF with a normal prior (dashed line). The dark, red, and green lines correspond to the logarithms of Bayes factors under observed effect sizes $\bar{x}/s = 0.4n$, $0.5n$, and $0.6n$, respectively.

difference is largest for b_{min} because this fraction is always smaller than b_{robust} and b_{freq}. Finally note that the AAFBFs show consistent behavior for this testing problem.

These two examples include the evaluation of equality constrained hypotheses in both continuous data and discrete data. Although the models used are simple, the results of the sensitivity study of adjusted fractional Bayes factors can be applied in the multivariate normal model where the parameters (e.g., the group means in ANOVA model, the coefficients in regression model) have a multivariate t distribution, and in multinomial model where the parameters (e.g., the probabilities in Contingency Tables) have a Dirichlet distribution which is the multivariate generalization of the Beta distribution. Furthermore, in more complicated settings such as structural equation models and generalized linear models, it can be anticipated that the larger b will result in less sensitive AFBFs because this implies that more data are used to specify the fractional prior such that the normal approximation to the prior has better performance based on the large sample theory.

Based on the discussion in this section, we propose the following scheme for specifying the fraction b in the AAFBF.

- Choose $b_{min} = (J+1)/n$ to have a default prior that is based on the idea of a minimal training sample.
- Choose $b_{robust} = \max\{(J+1)/n, \sqrt{n}/n\}$ to ensure that the default prior is close to normal.
- Choose $b_{freq} = \int_0^1 \exp(-n\beta_e^2/4)d\beta_e$ to control the frequentist error probabilities when testing an equality constrained hypothesis against the unconstrained alternative.

Note that n and J denote the sample size and the number of independent constraints for all the informative hypotheses, respectively.

5 Results for empirical examples

The examples introduced in Section 2 are revisited to illustrate how the AAFBF can be used to evaluate informative hypotheses. In the regression model, three parameters with respect to the regression coefficients are considered in the informative hypothesis $H_1: \theta_1 > 0, \theta_2 < 0, \theta_3 = 0$. The first step is to specify the prior and posterior distributions in (26) and (27), which needs the estimates $\hat{\boldsymbol{\theta}}$ and covariance matrix $\hat{\boldsymbol{\Sigma}}_\theta$ of the parameters. These can be obtained by analyzing the regression model with the data in Table 1

Table 3: Result for regression model example

	$b_{min} = 0.080$	$b_{robust} = 0.141$	$b_{freq} = 0.216$
$AAFBF_{11_c}$	6.04	4.46	3.55

using a number of statistical software (packages), such as Mplus (Muthén & Muthén, 2010) and R package lavaan (Rosseel, 2012). Note that we do not need to standardize the three coefficients as they are compared with zero. The analysis of data in lavaan renders the maximum likelihood estimates of the parameters, i.e., $\hat{\theta}_1 = 11.01$, $\hat{\theta}_2 = -2.85$, $\hat{\theta}_3 = -2.03$, and the covariance matrix:

$$\hat{\boldsymbol{\Sigma}}_\theta = \begin{bmatrix} 18.236 & -0.500 & 2.812 \\ -0.500 & 0.043 & -0.004 \\ 2.812 & -0.004 & 4.481 \end{bmatrix}.$$

To obtain the AAFBF for H_1 against H_{1_c}, the fraction b has to be specified. Based on the sample size of $n = 50$ and the length of vector $\boldsymbol{\beta}$ of $J = 3$ in this example, the three choices of fraction are $b_{min} = 0.080$, $b_{robust} = 0.141$, and $b_{freq} = 0.216$. Running `BaIn` with the estimates and covariance matrix of parameters of interest renders the AAFBF displayed in Table 3. As can be seen, $AAFBF_{11_c}$ is larger than 3 under each choice of b, which implies positive evidence in the data for H_1 against H_{1_c} according to Kass and Raftery (1995)'s rule.

The hypotheses in the repeated measures ANOVA model consists of four parameters of which the estimates are $\hat{\theta}_1 = 22.33$, $\hat{\theta}_2 = 22$, $\hat{\theta}_3 = 5.78$ and $\hat{\theta}_4 = 6.78$, and the covariance matrix is

$$\hat{\boldsymbol{\Sigma}}_\theta = \begin{bmatrix} 5.18 & 4.86 & 2.61 & 2.86 \\ 4.86 & 5.13 & 2.90 & 3.03 \\ 2.61 & 2.90 & 1.93 & 1.97 \\ 2.86 & 3.03 & 1.97 & 2.39 \end{bmatrix}.$$

Given sample size $n = 36$ and length of vector $\boldsymbol{\beta}$ of $J = 3$, three choices of b are automatically specified in `BaIn` as $b_{min} = 0.111$, $b_{robust} = 0.167$,

Table 4: Result for repeated measures ANOVA example

	$b_{min} = 0.111$	$b_{robust} = 0.167$	$b_{freq} = 0.255$
$AAFBF_{2u}$	4.60	3.07	2.01
$AAFBF_{2'u}$	0.24	0.24	0.24
$AAFBF_{22'}$	19.2	12.8	8.38

and $b_{freq} = 0.255$. Based on these specification BaIn renders the AAFBFs $AAFBF_{2u}$ for H_2 versus H_u and $AAFBF_{2'u}$ for $H_{2'}$ versus H_u. The results are shown in Table 4. As can be seen, $AAFBF_{2'u}$ is independent of b because the AAFBF for inequality constrained hypotheses is invariant to the choice of the fraction b. Thereafter, the AAFBF $AAFBF_{22'}$ for H_2 versus $H_{2'}$ can be computed by $AAFBF_{2u}/AAFBF_{2'u}$ which is shown in the last row in Table 4. The result of $AAFBF_{22'}$ in the last row suggests positive evidence in the data for H_2 against $H_{2'}$.

6 Conclusion

This paper presented a new approximate Bayesian procedure for the evaluation of informative hypotheses that can be used for virtually any model. The methodology is based on the prior adjusted default Bayes factor of Mulder (2014b). Furthermore, normal approximations were used to ensure fast computations. Numerical results showed that the approximation is close to the prior adjusted fractional Bayes factor. This implies that the proposed AAFBF provides an accurate quantification of the relative evidence between informative hypotheses. Furthermore, different choices were given for the fraction b, similar as in the fractional Bayes factor of O'Hagan (1995). The first choice relies on the concept of priors containing minimal information. The second choice uses a robustness argument resulting in a default prior distribution that is close to normal. The third choice is based on a frequency argument to control the classical error probabilities. The choice can be made by the user depending on the property which he/she finds most important. By computing the AAFBF for each choice of b we get a complete picture how much support there is in the data between two hypotheses when taking into account different philosophies.

We provide a software package BaIn with a user manual in the Appendix B to evaluate the informative hypotheses which only needs the maximum likelihood estimates and covariance matrix of the parameters of interest, denoted by $\boldsymbol{\theta}$ in this paper. BaIn computes the AAFBF for an informative hypothesis against an unconstrained hypothesis. By computing these quantifies for each informative hypothesis against the unconstrained hypothesis, psychological researchers can straightforwardly compute the relative support in the data for pairs of informative hypotheses.

The study in this paper contributes to the quantitative techniques in psychological research in three aspects. First, the proposed Bayesian test stimulates psychologists to translate scientific expectations to informative hypotheses that can be tested with the data in a direct manner. Second, the approximate Bayesian procedure allows psychologists to test their informative hypotheses in virtually any statistical model. Third, the software package allows psychologists to apply the new methodology on their own data in an easy manner.

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A Adjusting the prior mean for range constraints

The specification of the prior mean for $\boldsymbol{\beta}_1 = \boldsymbol{R}_{i_1}\boldsymbol{\theta} - \boldsymbol{r}_{i_1}$ in range constrained hypotheses consists of two steps:

1. Find the range constraints in the hypotheses under investigation. A hypothesis contains range constraint(s) if there exist lines in $\boldsymbol{R}_{i_1}$ of which the sum is zero vector. If there is more than one range constraint in the same hypothesis, then there are multiple sets of two or more lines that are added to zero. For example, hypothesis $H_1 : 0 < \theta_1 < \theta_2 < 1$ with $\boldsymbol{R}_{i_1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$ and $\boldsymbol{r}_{i_1} = (0, 0, -1)^T$ contains a range constraint, because $(\sum_{k=1}^{3} \boldsymbol{R}_{i_1}(k, 1), \sum_{k=1}^{3} \boldsymbol{R}_{i_1}(k, 2)) = (0, 0)$.

2. Specify the prior mean of $\boldsymbol{\beta}_1 = \boldsymbol{R}_{i_1}\boldsymbol{\theta} - \boldsymbol{r}_{i_1}$ for the range constraints. $\boldsymbol{\beta}_1$ contains the elements related to the range constraints and other inequality constraints. The prior means for those elements of $\boldsymbol{\beta}_1$ that represent the edges of a range constraint are specified as $\beta_1^* = -\sum_{k=1}^{K} r_{i_1}(k)/2$ where K is the number of lines in $\boldsymbol{R}_{i_1}$ for each range constraint and $r_{i_1}(k)$ is the constant for this range constraint, whereas the prior means for other elements of $\boldsymbol{\beta}_1$ are $\mathbf{0}$, which is not different from that for equality and inequality constrained hypotheses. For example, for hypothesis $H_1 : 0 < \theta_1 < \theta_2 < 1$ the edges of the range constraint are $\beta_{11} = \theta_1 > 0$ and $\beta_{13} = 1 - \theta_2 > 0$. Thus, β_{11} and β_{13} have prior means of 0.5, whereas $\beta_{12} = \theta_2 - \theta_1$ has a prior mean of 0.

B User manual of BaIn

The software package BaIn is developed in Fortran 90 with the IMSL 5.0 numerical library. It computes Bayes factors to evaluate any informative hypotheses (Section 2) and compare pairs from a set of informative hypotheses if they are comparable (Section 3.3). BaIn can be freely downloaded from the website `http://informative-hypotheses.sites.uu.nl/software/bain/`. The downloaded folder consists of an executable file "BaIn.exe", an input file "Input.txt", and an output file "Output.txt". Running "BaIn.exe" with "Input.txt" located in the same folder renders "Output.txt". This appendix instructs researchers to fill in the "Input.txt" such that "BaIn.exe" can properly read the information. The "Input.txt" mainly contains the estimates and covariance matrix of parameters $\boldsymbol{\theta}$ for prior and posterior specification, and the restriction matrix and constant vector for each informative hypothesis.

The repeated measures ANOVA example in Section 2.2 is used to illustrate the valid specification of input file. We will first display and then explain the context below written in the "Input.txt" when evaluating informative hypothesis H_2 (11) and $H_{2'}$ (12).

```
#Number of parameters of interest; Number of informative hypotheses; Sample size
4 2 36
#Estimates of parameters
22.33 22 5.78 6.78
#Covariance matrix of parameters
5.18 4.86 2.61 2.86
4.86 5.13 2.90 3.03
2.61 2.90 1.93 1.97
2.86 3.03 1.97 2.39
#Numbers of equality and inequality constraints in H1
2 1
#Restriction matrix (R0|r0) for equality constraints
1 -1  0  0   0
0  0  1 -1   0
#Restriction matrix (R1|r1) for inequality constraints
0  1 -1  0   0
#Numbers of equality and inequality constraints in H2
0 3
#Restriction matrix (R0|r0) for equality constraints
#Restriction matrix (R1|r1) for inequality constraints
1 -1  0  0   0
```

```
22   0  1 -1  0   0
23   0  0  1 -1   0
```

The input text has strictly fixed structure. There are annotation lines starting with `#` below which the corresponding information (numbers) has to be given. The first line is the annotation for the number of structural parameters, number of informative hypotheses, and sample size, which means we need to write three numbers in the second line, i.e., 4, 2 and 9. Because the number of structural parameters is 4, four numbers for the estimates of parameters are presented in line 4, and a 4×4 covariance matrix is written in lines 6 to 9. Furthermore, because the number of informative hypotheses is 2, two hypotheses are specified. For the first hypothesis, line 11 specifies 2 and 1 for the numbers of equality and inequality constraints, respectively. Therefore, the augmented restriction matrix with constant vector for equality constraints has two rows shown in lines 13 and 14, and one row for inequality constraints in line 16. For the second hypothesis, the numbers of equality and inequality constraints are 0 and 3 given in line 18, respectively. As can be seen, there is not a line with numbers below the annotation line 19 `#Restriction matrix (R0|r0) for equality constraints` because this hypothesis does not contain any equality constraint. While from lines 21 to 23 the augmented restriction matrix for three inequality constraints are displayed.

The estimates and covariance matrix of structural parameters can be obtained from other statistical software, e.g., Mplus (Muthén & Muthén, 2010) and R package lavaan (Rosseel, 2012), and the augment restriction matrix `(R0|r0)` and `(R1|r1)` can be specified based on the informative hypotheses under evaluation. Executing "BaIn.exe" with these information renders the relative complexities, fits, and Bayes factors for informative hypotheses under different choices of fraction b in the "Output.txt". The results for repeated measures ANOVA example is shown as follows.

```
Result for H1

Equality constraints
   Fit    Complexity (b1)   Complexity (b2)   Complexity (b3)
  0.091         0.049              0.059              0.096

Inequality constraints (conditional on equality constraints)
   Fit    Complexity (b1)   Complexity (b2)   Complexity (b3)
  1.000         0.500              0.500              0.500
```

```
Number of iterations
  3000          3000              3000              3000

BF1u (b1=0.111)    BF1u (b2=0.167)    BF1u (b3=0.255)
     4.603              3.069              2.006

BF1c (b1=0.111)    BF1c (b2=0.167)    BF1c (b3=0.255)
     4.603              3.069              2.006


Result for H2

Equality constraints
   Fit    Complexity (b1)   Complexity (b2)   Complexity (b3)
  1.000        1.000             1.000            1.000

Inequality constraints (conditional on equality constraints)
   Fit    Complexity (b1)   Complexity (b2)   Complexity (b3)
  0.023        0.096             0.098            0.097
Number of iterations
  46000         9000              9000             9000

BF2u (b1=0.111)    BF2u (b2=0.167)    BF2u (b3=0.255)
     0.240              0.237              0.238

BF2c (b1=0.111)    BF2c (b2=0.167)    BF2c (b3=0.255)
     0.223              0.219              0.220
```

The results contain the relative fits and complexities for both equality and inequality constraints, as well as the Bayes factors under different fraction b in each hypothesis. For equality constraint, the relative fit and complexity are the normal posterior and prior densities in (28), and thus can be directly computed. However, the computation of relative fit and complexity for inequality constraints is often difficult and needs to sample from the posterior and prior distributions using Monte Carlo Markov Chain methods (Gu et al., 2014). BaIn uses an efficient algorithm, which requires less number of iterations (displayed below fit and complexities) in the Markov chains to accurately estimate the relative fit and complexity. Note that the Bayes factor for informative hypotheses H_1 against H_2 can be computed using (34) with BF_{1u} and BF_{2u}.