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Computing Bayes Factors from Data with Missing Values

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Abstract

The Bayes factor is increasingly used for the evaluation of hypotheses. These may be traditional hypotheses specified using equality constraints among the parameters of the statistical model of interest or informative hypotheses specified using equality and inequality constraints. So far no attention has been given to the computation of Bayes factors from data with missing values. A key property of such a Bayes factor should be that it is only based on the information in the observed values. This paper will show that such a Bayes factor can be obtained using multiple imputations of the missing values.

After introduction of the general framework elaborations for Bayes factors based on default or subjective prior distributions and Bayes factors based on priors specified using training data will be given. It will be illustrated that the approach proposed can be applied using R packages for multiple imputation in combination with the Bayes factor packages `Bain` and `BayesFactor`. It will furthermore be illustrated that Bayes factors computed using a single imputation of the data are very inaccurate approximations of the correct Bayes factor.

Keywords: Bayes Factor, Informative Hypotheses, Missing Data, Multiple Imputation.

Computing Bayes Factors from Data with Missing Values

Introduction

The Bayes factor (Kass and Raftery, 1995; Mulder and Wagenmakers, 2016) is increasingly used for the evaluation of traditional hypotheses specified using equality constraints among the parameters of the statistical model of interest and informative hypotheses (Hojtink, 2012) specified using inequality and equality constraints (see for applications, for example, Van Schie et al., 2016, and Kolkman et al., 2013). This paper will clarify how these hypotheses can be evaluated using Bayes factors computed from data with missing values. First of all, multiple imputation of the missing values will be used to create multiple completed data matrices (Rubin, 1987; Shafer, 1997, Van Buuren, 2012). Secondly, it will be shown that these completed data matrices can be used to approximate the posterior and prior distributions of the parameters of the model at hand such that only the information in the observed values and not the information in the imputed values, is used. Finally, it will be shown that the Bayes factor using only the information in the observed values is a function of these prior and posterior distributions. A strong point of this approach is that each completed data matrix can easily be analyzed using software (e.g., for parameter estimation or Bayes factor computation) that requires data without missing values, and, as will be elaborated in this paper, subsequent combination of these analyses renders Bayes factors using only the information in the observed values and not the information in the imputed values.

The focus will be on two approaches for which software implementations are readily available. The first combines multiple imputation with the evaluation of informative hypotheses using the approximate adjusted fractional Bayes factor (Mulder, 2014; Gu et al., 2014; Gu, 2016; Gu et al. 2016; Gu, Mulder, and Hoijtink, in press) as implemented in the R package `Bain` (<https://informative-hypotheses.sites.uu.nl/software/bain/>). The second combines multiple imputation with the g-prior based Bayes factors implemented in the `BayesFactor` package

(<http://bayesfactorpcl.r-forge.r-project.org/>). This Bayes factor is mainly suited for the evaluation of hypotheses specified using equality constraints (see, for example, Rouder et al., 2009) although there are options to test one-sided hypotheses and about equality constraints (Morey and Rouder, 2011). More information about the models that can be handled with both packages will be provided in the section about informative hypotheses. It will be argued that the combination of multiple imputation with `Bain` or `BayesFactor` provides a versatile tool for the the evaluation of informative hypotheses if the data contain missing values. There are many R packages that can be used to create multiple imputations (see, <http://stefvanbuuren.nl/mi/Software.html>, for an overview). This paper will be illustrated using the R packages `norm` (Shafer, 1997) which uses, so called, joint multiple imputation, that is, "it is assumed that the data follow a known joint probability distribution with unknown parameters" (Kropko et al., 2014) and `mice` (<http://www.stefvanbuuren.nl/mi/mice.html>; Van Buuren, 2012) which, like the `mi` package (<https://cran.r-project.org/web/packages/mi/index.html>), uses, so called, full conditional imputation, that is, only the distribution of each variable conditional upon the other variables is specified. The differences between and options provided by both approaches will be highlighted later in this paper. To ensure applicability of the approaches presented in this paper, the data and annotated codes used for each of the examples are made available as online supplementary materials.

Additionally, in Appendix A, there is attention for Bayes factors that can currently not easily be applied in the context of missing data. The fractional Bayes factor (O'Hagan, 1995) can be used for the evaluation of hypotheses specified using equality constraints. It will be presented but not illustrated because it is not implemented in software. Two Bayes factors using minimal training samples (Perez and Berger, 2002; Berger and Pericchi, 1996, 2004) to specify the prior distribution will be presented. The first can be used to evaluate hypotheses specified using equality constraints. The second, implemented in the software package `BIEMS` (Mulder, Hoijtink, and de Leeuw, 2012), can be used for the evaluation of

hypotheses specified using equality and inequality constraints. For both it will be shown how they can be computed from data with missing values. This will not be illustrated because for the first no software exists and BIEMS does not provide the information necessary to compute the Bayes factor from data with missing values. The paper is concluded with a short discussion.

Analysis Model and Imputation Model

In this section the analysis model and the imputation model will be introduced (Meng, 1995; Schafer, 1999). Here and in the following sections the elaborations will be general and illustrated using a normal linear regression model.

To make inferences from data, an analysis model has to be specified. This can be, for example, a normal linear regression model, a structural equation model, or a multilevel model. The data will be denoted by \mathbf{X} . Often \mathbf{X} is a $N \times P$ matrix, where N denotes the number of persons and P the number of variables. In many cases \mathbf{X} contains missing values, that is, \mathbf{X} can be split into \mathbf{X}_m and \mathbf{X}_o , where \mathbf{X}_o contains the observed values and \mathbf{X}_m the missing values. The density of the data of the analysis model will be denoted by $p(\mathbf{X} \mid \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ denotes the parameters of the analysis model. These can be, for example, means, regression coefficients, or factor loadings. In the next section, the model parameters will be decomposed into $\boldsymbol{\theta} = [\boldsymbol{\gamma}, \boldsymbol{\omega}]$ where $\boldsymbol{\gamma}$ denotes the parameters with respect to which hypotheses are formulated, and $\boldsymbol{\omega}$ the nuisance parameters.

Example 1: Normal Linear Regression

Let $\mathbf{X} = [\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2]$, with $\mathbf{y} = [y_1, \dots, y_i, \dots, y_N]$, $\mathbf{x}_1 = [x_{11}, \dots, x_{i1}, \dots, x_{N1}]$, and $\mathbf{x}_2 = [x_{12}, \dots, x_{i2}, \dots, x_{N2}]$ where each variable is continuous. The normal linear regression model is

$$y_i = \alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2} + e_i, \text{ and } x_{i1}, x_{i2} \sim \mathcal{N}(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x) \quad (1)$$

where $i = 1, \dots, N$, α_0 denotes the intercept, α_1 and α_2 relate the continuous predictors \mathbf{x}_1

and \mathbf{x}_2 to the dependent variable \mathbf{y} , $e_i \sim \mathcal{N}(0, \sigma^2)$ denotes the error in prediction, and $\boldsymbol{\mu}_x$ and $\boldsymbol{\Sigma}_x$ denote the mean vector and covariance matrix of \mathbf{x}_1 and \mathbf{x}_2 , respectively. The density of the data is

$$p(\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2 \mid \boldsymbol{\alpha}, \sigma^2, \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x) = p(\mathbf{y} \mid \mathbf{x}_1, \mathbf{x}_2, \boldsymbol{\alpha}, \sigma^2) p(\mathbf{x}_1, \mathbf{x}_2 \mid \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x) =$$

$$\frac{1}{\sqrt{2\pi\sigma^2}^N} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^N (y_i - \alpha_0 - \alpha_1 x_{i1} - \alpha_2 x_{i2})^2\right) \mathcal{N}(\mathbf{x}_1, \mathbf{x}_2 \mid \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x), \quad (2)$$

in which a model is specified for the predictors too, that is, they are not assumed to be fixed. In this example $\boldsymbol{\gamma} = [\alpha_1, \alpha_2]$ because these are the parameters with respect to which hypotheses will be formulated in the next section. Consequently, the nuisance parameters are $\boldsymbol{\omega} = [\alpha_0, \sigma^2, \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x]$.

End Example 1

When data contain missing values, $p(\mathbf{X}_m, \mathbf{X}_o \mid \boldsymbol{\theta})$ can not straightforwardly be evaluated. This problem can be solved using multiple imputation (Rubin, 1987; Shafer, 1997; van Buuren, 2012). Classical applications of multiple imputation consist of three steps. In the first step an appropriately chosen imputation model is used to create multiple completed data matrices in which the missing values are imputed. In the second step each completed data matrix is used to estimate the parameters of the analysis model. In the third step, the Rubin rules (see, for example, van Buuren, 2012, pp. 37-38) are used to combine these estimates and their covariance matrix such that the resulting overall estimates and covariance matrix are based only on the information in the observed values and not on the information in the imputed values.

As will be elaborated later in this paper and Appendix A, steps two and three are only used when the Bayes factor implemented in `Bain` is computed from data with missing values. However, for all Bayes factors an imputation model has to be used to create completed data matrices. The imputation model will be denoted by

$p(\mathbf{X}_m, \mathbf{X}_o, \mathbf{Z}_m, \mathbf{Z}_o \mid \boldsymbol{\theta}, \boldsymbol{\eta})$. Note that \mathbf{Z} is a $N \times R$ matrix containing auxiliary variables. Analogous to \mathbf{X} it can be split into parts containing the observed and missing values: \mathbf{Z}_o and \mathbf{Z}_m . Note furthermore, that the parameters of the imputation model consist of the parameters $\boldsymbol{\theta}$ of the analysis model augmented with parameters $\boldsymbol{\eta}$ that account for the presence of the auxiliary variables \mathbf{Z} . Why this is a necessary requirement will be elaborated when simplifying Equation 16 to 18 and Equation 20 to 22.

The imputation model has to be chosen such that conditional on the variables in this model the missing data are believed to be missing at random (MAR, van Buuren, 2012, pp. 6-8, 31-33), that is, the distribution of missingness does not depend on $\mathbf{X}_m, \mathbf{Z}_m$ (Shafer and Graham, 2002). The big question is which auxiliary variables \mathbf{Z} have to be added to the variables \mathbf{X} already in the analysis model to achieve MAR. Unless missingness is planned the mechanism causing the missingness is unknown to the researcher. Shafer and Graham (2002) write "When missingness is beyond the researchers's control, its distribution is unknown and MAR is only an assumption. In general, there is no way to test whether MAR holds in a data set". They also write "Although it is not necessary to have a scientific theory underlying an imputation model, it is crucial for that model to be general enough to preserve effects of interest in later analyses." What is often done in practice is that the variables \mathbf{X} from the analysis model are augmented with auxiliary variables \mathbf{Z} that are expected to be good predictors of the variables in the analysis model containing missing values. Although there can be a convincing argument with respect to which and how many auxiliary variables to add, there is no way to test whether MAR is achieved, MAR is only an assumption. This holds for classical applications of multiple imputation and also when it is used to compute Bayes factor from data with missing values. For a comprehensive overview of multiple imputation and construction of the imputation model the interested reader is referred to the references given in this section. Their elaborations directly apply to the use of multiple imputation when the goal is to compute Bayes factor from data with missing values.

After the choice of \mathbf{X} and \mathbf{Z} , the imputation model $p(\mathbf{X}_m, \mathbf{X}_o, \mathbf{Z}_m, \mathbf{Z}_o \mid \boldsymbol{\theta}, \boldsymbol{\eta})$ has to be specified. There are two dominant approaches. The first is joint modeling, that is, a complete specification of $p(\mathbf{X}_m, \mathbf{X}_o, \mathbf{Z}_m, \mathbf{Z}_o \mid \boldsymbol{\theta}, \boldsymbol{\eta})$ as is used in Shafer (1997) and the corresponding R packages `norm` in which \mathbf{X} and \mathbf{Z} are modeled using a multivariate normal distribution (Chapters 5 and 6), `cat` in which \mathbf{X} and \mathbf{Z} are modeled using a log-linear model (Chapters 7 and 8), and `mix` in which \mathbf{X} and \mathbf{Z} are modeled using a general location model (Chapter 9). The second is full conditional specification (van Buuren, 2012) in which only the conditional density of each variable given the other variables is specified as is implemented in the R packages `mice` and `mi`.

The statistical theory underlying joint modeling is crystal clear: sample $\mathbf{X}_m, \mathbf{Z}_m, \boldsymbol{\theta}$, and $\boldsymbol{\eta}$ iteratively from a posterior distribution based on $p(\cdot)$ and use the sampled values of $\mathbf{X}_m, \mathbf{Z}_m$ to create completed data matrices. However, joint modeling may be difficult or impossible in a case of complex data structures (e.g., data requiring multilevel modeling) or when the data not only contain groups and continuous dependent and predictor variables, a situation which is covered by the general location model (Shafer, 1997, Chapter 9), but also variables requiring modeling with, for example, a binomial or Poisson distribution. In contrast, full conditional specification is usually easy and straightforward, but until recently the statistical theory underlying this approach was only poorly understood. However, meanwhile it has become clear that when the conditional specifications are linear, logistic, and multinomial regressions, these are compatible with a restricted general location model (Hughes et al., 2014; Liu, et al., 2014; Seaman and Hughes, 2016). Furthermore, there is ample evidence that full conditional specification renders proper imputations even even there is no compatible joint distribution. The interested reader is referred to Kropko et al. (2014), Liu et al. (2014), van Buuren et al. (2007), van Buuren (2007), and Zhu and Raghunathan (2015) for further elaborations.

As will be elaborated in the section "Imputation of Missing Values", for the developments in this paper it is irrelevant whether joint modeling or full conditional

modeling is used. It is also irrelevant which of the available R packages is used. The illustrations given will be executed with the R packages `norm` and `mice`, that is, one using joint modeling and one using full conditional specification. This will provide a stepping stone for researchers wanting to use other packages of either type.

Example 1 Continued: Normal Linear Regression

When there is one auxiliary variable, that is, $\mathbf{Z} = \mathbf{z}$, with $\mathbf{z} = [z_1, \dots, z_i, \dots, z_N]$, a common imputation model $p(\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{z} \mid \boldsymbol{\theta}, \boldsymbol{\eta})$ in the context of normal linear regression is:

$$\begin{pmatrix} \mathbf{y} \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{z} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} m_y \\ m_1 \\ m_2 \\ m_z \end{pmatrix}, \begin{pmatrix} s_y^2 & s_{y1} & s_{y2} & s_{yz} \\ s_{y1} & s_1^2 & s_{12} & s_{1z} \\ s_{y2} & s_{12} & s_2^2 & s_{2z} \\ s_{yz} & s_{1z} & s_{2z} & s_z^2 \end{pmatrix} \right), \quad (3)$$

(Schafer, 1997, p. 157; Van Buuren, 2012, pp. 105-107) where, $\boldsymbol{\eta} = [m_z, s_{1z}, s_{2z}, s_z^2, s_{yz}]$ and $\boldsymbol{\theta} = [m_y, m_1, m_2, s_y^2, s_{y1}, s_{y2}, s_1^2, s_{12}, s_2^2]$. Note that, $\boldsymbol{\alpha}, \sigma^2, \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x$ are a function of $\boldsymbol{\theta}$, and therefore that the imputation model encompasses the analysis model presented in Equation 1. Note furthermore, as will be highlighted later on, that Equation 3 can be implemented using both joint modeling and full conditional modeling (Hughes et al., 2014).

End Example 1

Informative Hypotheses

An important tool in psychological research is the evaluation of hypotheses with respect to the parameters of the analysis model. In this paper we consider the class of linear (in)equality constrained hypotheses also known as informative hypotheses (Hojtink, 2012, Chapter 1). If $\boldsymbol{\theta} = [\boldsymbol{\gamma}, \boldsymbol{\omega}]$, where $\boldsymbol{\gamma}$ is vector of length J containing the parameters that are used in the hypotheses and $\boldsymbol{\omega}$ contains the parameters that are not used, for

$k = 1, \dots, K$ the hypotheses can be formalized as:

$$H_k : \mathbf{S}_k \boldsymbol{\gamma} = \mathbf{s}_k, \mathbf{R}_k \boldsymbol{\gamma} > \mathbf{r}_k, \quad (4)$$

where \mathbf{S}_k is a $J \times k_1$ matrix imposing k_1 equality constraints on $\boldsymbol{\gamma}$, \mathbf{R}_k is a $J \times k_2$ matrix imposing k_2 inequality constraints, and \mathbf{s}_k and \mathbf{r}_k are vectors containing constants of size k_1 and k_2 , respectively. Additionally of interest is $H_u : \boldsymbol{\gamma}$, that is, $H_u : \gamma_1, \dots, \gamma_J$, an hypothesis without constraints on the parameters $\boldsymbol{\gamma}$. This definition of H_u will be used throughout this paper. If H_1 through H_K are formulated with respect to the parameters collected in $\boldsymbol{\gamma}$, then H_u is the hypotheses that states that there are no restrictions on $\boldsymbol{\gamma}$. Consequently, H_1 through H_K are always nested within H_u . As will be shown in the next section, this property is used to represent the Bayes factor in terms of the fit and the complexity of the hypotheses entertained.

The persistent critique on p-values (see, for example, Wagenmakers, 2007, and Cumming, 2012) has led to an increased attention for the Bayes factor as a tool for hypothesis evaluation. The R package `BayesFactor` allows researchers to evaluate classical null-hypotheses using the Bayes factor in the context of the normal linear model (e.g., ANOVA, multiple regression, t-tests) and contingency tables. However, also the null-hypothesis has been criticized. Both Cohen (1994) and Royal (1997) criticize the null-hypothesis for being so specific that it is often hard to imagine a population where it holds. This is nicely summarized in Cohen's (1994) title "The earth is round, $p < .05$ ", that is, a precise hypothesis can be rejected without using data. Another critique is that the null-hypothesis usually does not represent the theory or expectation a researcher has. As is highlighted in Hoijtink (2012) (see also <https://informative-hypotheses.sites.uu.nl/>), informative hypotheses are an extension of the classical null-hypothesis (see Equation 4) that can represent the theory or expectation of a researcher. Simple examples are: "The depression level after therapy (μ_1) is smaller than the depression level after medication (μ_2) which in turn is smaller than the depression level in a control group (μ_3)" leading to $H_k : \mu_1 < \mu_2 < \mu_3$; "The larger

intelligence and social economic status (with standardized regression coefficients β_1 and β_2 , respectively), the larger income, but intelligence is the stronger predictor" leading to $H_k : \beta_1 > 0, \beta_2 > 0, \beta_1 > \beta_2$; and, "the effect of the new medication on headache (μ_1) is irrelevantly different from the effect of the old medication (μ_2)" leading to $H_k : |\mu_1 - \mu_2| < d \times s$, where s is an estimate of the within group variance and d denotes Cohen's d for which a value demarcating (ir)relevance can be chosen by the researcher. Informative hypotheses are increasingly used by psychological researchers. An overview can be found at

<https://informative-hypotheses.sites.uu.nl/publications/applications/>.

Classical hypotheses and informative hypotheses can be evaluated using the Bayes factor implemented in **Bain**. It has thus far (see, Gu, 2016) been applied in the context of the multivariate normal linear model, logistic regression, multilevel modeling, confirmatory factor analysis, and structural equation modeling. It can be applied to any model for which a normal approximation of the posterior distribution of the model parameters is reasonable.

Example 1 Continued: Normal Linear Regression

For the regression model displayed in Equation 1 $\gamma = [\alpha_1, \alpha_2]$, that is, hypotheses are formulated with respect to both regression coefficients but not with respect to the other parameters. Three hypotheses will be used that are specifications of Equation 4:

$$H_1 : \alpha_1 = \alpha_2 = 0, \quad (5)$$

$$H_2 : \alpha_1 > 0, \alpha_2 > 0, \alpha_1 > \alpha_2, \quad (6)$$

and,

$$H_u : \alpha_1, \alpha_2, \quad (7)$$

HIGHLIGHT DE ROL VAN Hu where H_1 specifies that both regression coefficients are zero and H_2 specifies that both regression coefficients are larger than zero and that the first is larger than the second. Note that the latter restriction only makes sense if the scale on

which both predictors are measured is considered to be comparable. As can be seen, H_1 and H_2 specify hypotheses with respect to α_1 and α_2 , therefore, H_u is the hypothesis in which there are no restrictions on both regression coefficients: $H_u : \alpha_1, \alpha_2$. Note that, for H_2

$$\mathbf{R}_2 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \quad (8)$$

and $\mathbf{r}_2 = -[0, 0]$. Only the last two restrictions in Equation 6 are specified because they render the first redundant.

End Example 1

Bayes Factor for the Evaluation of Informative Hypotheses in the Absence of Missing Values

The basic structure of the Bayes factor implemented in `BayesFactor`, `Bain` and other types of Bayes factors (see Appendix A) used for the evaluation of hypotheses of the form displayed in Equation 4 is identical. Each requires the specification of the posterior and prior distribution of the parameters of the model at hand. The posterior distribution will be denoted by $g_u(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o)$, where the subscript u denotes that it is the posterior distribution under H_u . Similarly, the prior distribution will be denoted by $h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o)$.

Both the mean and the (co)variance(matrix) of the prior distribution require careful specification and will now shortly be discussed. When testing the hypothesis $H_0 : \gamma = 0$ versus $H_u : \gamma \neq 0$ with data $x_i \sim \mathcal{N}(\gamma, \omega)$, for $i = 1, \dots, N$, Jeffreys (1961) suggests to use $h_u(\gamma, \omega) = h_u(\gamma)h_u(\omega)$ and to give $h_u(\gamma)$ a mean of zero, that is, centered on the null-value used in H_0 . Inspired by Jeffreys the ANOVA, regression, and t-test functions implemented in `BayesFactor` also center the prior distribution on the values specified by the null hypothesis, see, for example, Rouder et al. (2009). Mulder (2014) generalizes this principle to hypotheses of the general form displayed in Equation 4 using, what he calls, an

adjustment of the prior mean that can be obtained as follows. Let

$$\gamma_B \in \{S_k \gamma_B = s_k, R_k \gamma_B = r_k\} \text{ for } k = 1, \dots, K, \quad (9)$$

that is, γ_B denotes a value of γ on the boundary of all the hypotheses under investigation.

Let $\hat{\gamma}$ denote the mean of $h_u(\gamma, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o)$ with respect to γ . Then using

$\gamma_{adj} = -\hat{\gamma} + \gamma_B$ ensures that the mean of $h_u(\gamma + \gamma_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o)$ is equal to γ_B .

Applied to H_0 this adjustment will render a prior mean of 0 and applied to the ANOVA, regression, and t-test functions implemented in `BayesFactor` it will also render the values specified by the respective null hypotheses. Often the parametric shape of $h_u(\gamma)$ is normal, Cauchy, or t. This implies that also the prior (co)variance(matrix) or scale(matrix) has to be specified. As will be elaborated later in this paper, for the Bayes factors implemented in `Bain` and discussed in Appendix A this is done using the a fraction of the information in the data, for the Bayes factors implemented in `BayesFactor` a subjective specification that is independent of the data is required.

As can be seen in, for example, Hoijtink (2012, pp. 51-52), the Bayes factor is often used to compare H_k with H_u . Based on Chib's (1995) representation of the Bayes factor, and the derivation presented in Hoijtink (2012, p. 59, see also Appendix B of this paper), the Bayes factor comparing the informative hypothesis H_k with the unconstrained hypothesis H_u can be written as:

$$BF_{ku} = \frac{f_k}{c_k}, \quad (10)$$

where f_k denotes the fit of H_k , that is,

$$f_k = \int_{\boldsymbol{\theta} \in H_k} \int_{\boldsymbol{\eta}} g_u(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) d\boldsymbol{\theta} d\boldsymbol{\eta}. \quad (11)$$

If hypotheses are specified using only inequality constraints, this is the proportion of the posterior distribution $g_u(\cdot)$ in agreement with H_k . The complexity of H_k is denoted by c_k , that is,

$$c_k = \int_{\gamma \in H_k} \int_{\boldsymbol{\omega}} \int_{\boldsymbol{\eta}} h_u(\gamma + \gamma_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) d\gamma d\boldsymbol{\omega} d\boldsymbol{\eta}. \quad (12)$$

If hypotheses are specified using only inequality constraints, this is the proportion of the prior distribution $h_u(\cdot)$ in agreement with H_k . In the absence of missing data, after specification of $g_u(\cdot)$ and $h_u(\cdot)$, the Bayes factor displayed in Equation (10) can straightforwardly be computed using the packages `BayesFactor` and `Bain`. The Bayes factor BF_{ku} quantifies the relative support in the data for H_k and H_u . If, for example, $BF_{ku} = 5$ the support for H_k is five times stronger than the support for H_u . From the Bayes factors versus the unconstrained model the Bayes factor of H_k versus $H_{k'}$ can be obtained using $BF_{kk'} = BF_{ku}/BF_{k'u}$.

In general it is not easy to compute Bayes factors. However, the interested reader is referred to Chib (1995) and Chib and Jeliazkov (2001) for a generally applicable procedure that gives stable results. However, these procedures do not have to be applied when Bayes factors are computed using `BayesFactor` or `Bain`. The Bayes factors implemented in `BayesFactor` can exactly be computed via the evaluation of relatively simple formulas (see, Rouder et al., 2009, for an example). The Bayes factors implemented in `Bain` continues a tradition started by Klugkist, Laudy, and Hoijtink (2005) who estimated f_k and c_k by counting the proportion of samples from the posterior and prior distributions, respectively, in agreement with H_k . This idea was improved upon by Mulder, Hoijtink, and de Leeuw (2012) who decomposed the fit into one component for each of the $k_1 + k_2$ constraints used to specify H_k such that very accurate estimates of each component could be obtained even if the number of components is large. This idea was further improved by Gu (2016) and Gu, Mulder, and Hoijtink (in press) who simplified the computation of the fit of the k_1 components specified using equality constraints and increased the speed of computation without sacrificing accuracy.

Bayes Factor for the Evaluation of Informative Hypotheses from Data with Missing Values

To obtain the Bayes factor from Equation 10 from data with missing values, f_k and c_k , that is, the fit and complexity computed using only the information in the observed values and not the imputed values, have to be computed. The derivation of the fit from data with missing values starts with the observation that

$$\begin{aligned} g_u(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) &= \int_{\mathbf{X}_m, \mathbf{Z}_m} g_u(\boldsymbol{\theta}, \boldsymbol{\eta}, \mathbf{X}_m, \mathbf{Z}_m \mid \mathbf{X}_o, \mathbf{Z}_o) d\mathbf{X}_m d\mathbf{Z}_m = \\ &\int_{\mathbf{X}_m, \mathbf{Z}_m} g_u(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_m, \mathbf{Z}_m, \mathbf{X}_o, \mathbf{Z}_o) g(\mathbf{X}_m, \mathbf{Z}_m \mid \mathbf{X}_o, \mathbf{Z}_o) d\mathbf{X}_m d\mathbf{Z}_m \approx \\ &\frac{1}{Q} \sum_{q=1}^Q g_u(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_m^q, \mathbf{Z}_m^q, \mathbf{X}_o, \mathbf{Z}_o), \end{aligned} \quad (13)$$

where $\mathbf{X}_m^q, \mathbf{Z}_m^q$ denotes the q -th imputation of the missing values obtained by sampling from $g(\mathbf{X}_m, \mathbf{Z}_m \mid \mathbf{X}_o, \mathbf{Z}_o) = \int_{\boldsymbol{\theta}} \int_{\boldsymbol{\eta}} g_u(\boldsymbol{\theta}, \boldsymbol{\eta}, \mathbf{X}_m, \mathbf{Z}_m \mid \mathbf{X}_o, \mathbf{Z}_o) d\boldsymbol{\eta} d\boldsymbol{\theta}$. Note that the latter can be achieved by iteratively sampling from

$$g_u(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_m, \mathbf{Z}_m, \mathbf{X}_o, \mathbf{Z}_o) \quad (14)$$

and

$$p(\mathbf{X}_m, \mathbf{Z}_m \mid \boldsymbol{\theta}, \boldsymbol{\eta}, \mathbf{X}_o, \mathbf{Z}_o), \quad (15)$$

and only retaining the sampled values \mathbf{X}_m^q and \mathbf{Z}_m^q at each iteration q . Using Equation 13, the fit can be computed from data with missing values using

$$f_k = \int_{\boldsymbol{\theta} \in H_k} \int_{\boldsymbol{\eta}} \frac{1}{Q} \sum_{q=1}^Q g_u(\boldsymbol{\theta} \mid \mathbf{X}_m^q, \mathbf{Z}_m^q, \mathbf{X}_o, \mathbf{Z}_o) g_u(\boldsymbol{\eta} \mid \boldsymbol{\theta}, \mathbf{X}_m^q, \mathbf{Z}_m^q, \mathbf{X}_o, \mathbf{Z}_o) d\boldsymbol{\eta} d\boldsymbol{\theta}. \quad (16)$$

This result can be simplified to

$$f_k = \int_{\boldsymbol{\theta} \in H_k} \frac{1}{Q} \sum_{q=1}^Q g_u(\boldsymbol{\theta} \mid \mathbf{X}_m^q, \mathbf{X}_o) d\boldsymbol{\theta} = \frac{1}{Q} \sum_{q=1}^Q f_k^q, \quad (17)$$

or in terms of $\boldsymbol{\gamma}$

$$f_k = \int_{\boldsymbol{\gamma} \in H_k} \frac{1}{Q} \sum_{q=1}^Q g_u(\boldsymbol{\gamma} \mid \mathbf{X}_m^q, \mathbf{X}_o) d\boldsymbol{\gamma} = \frac{1}{Q} \sum_{q=1}^Q f_k^q, \quad (18)$$

where f_k^q denotes the fit computed from the q th imputed data matrix (cf. Equation 11). As can be seen, the fit computed from data with missing values is the average of the fits computed for Q imputed data matrices. Note that, if H_k specifies that one or more of the γ 's has a fixed value (e.g., $\gamma_1 = 0$) the integration in Equations 17 and 18 evaluates $g_u(\cdot)$ with these γ 's fixed at the specified values (e.g., $g_u(\gamma_1 = 0, \gamma_2, \dots, \gamma_J | \mathbf{X}_m^q, \mathbf{X}_o)$ integrated with respect to $\gamma_2, \dots, \gamma_J$). The same holds for the integration in Equations 21 and 22 below.

Similarly, the derivation of the complexity starts with the observation that

$$\begin{aligned} h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} | \mathbf{X}_o, \mathbf{Z}_o) &= \int_{\mathbf{X}_m, \mathbf{Z}_m} h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta}, \mathbf{X}_m, \mathbf{Z}_m | \mathbf{X}_o, \mathbf{Z}_o) d\mathbf{X}_m d\mathbf{Z}_m = \\ &= \int_{\mathbf{X}_m, \mathbf{Z}_m} h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} | \mathbf{X}_m, \mathbf{Z}_m, \mathbf{X}_o, \mathbf{Z}_o) g(\mathbf{X}_m, \mathbf{Z}_m | \mathbf{X}_o, \mathbf{Z}_o) d\mathbf{X}_m d\mathbf{Z}_m = \\ &= \frac{1}{Q} \sum_{q=1}^Q h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} | \mathbf{X}_m^q, \mathbf{Z}_m^q, \mathbf{X}_o, \mathbf{Z}_o). \end{aligned} \quad (19)$$

Subsequently, using analogous steps as for the fit, the complexity can be computed as:

$$c_k = \int_{\boldsymbol{\gamma} \in H_k} \int_{\boldsymbol{\omega}} \int_{\boldsymbol{\eta}} \frac{1}{Q} \sum_{q=1}^Q h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} | \mathbf{X}_m^q, \mathbf{Z}_m^q, \mathbf{X}_o, \mathbf{Z}_o) d\boldsymbol{\gamma} d\boldsymbol{\omega} d\boldsymbol{\eta}, \quad (20)$$

which can be simplified to

$$c_k = \int_{\boldsymbol{\gamma} \in H_k} \int_{\boldsymbol{\omega}} \frac{1}{Q} \sum_{q=1}^Q h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega} | \mathbf{X}_m^q, \mathbf{X}_o) d\boldsymbol{\gamma} d\boldsymbol{\omega} = \frac{1}{Q} \sum_{q=1}^Q c_k^q, \quad (21)$$

or in terms of $\boldsymbol{\gamma}$

$$c_k = \int_{\boldsymbol{\gamma} \in H_k} \frac{1}{Q} \sum_{q=1}^Q h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj} | \mathbf{X}_m^q, \mathbf{X}_o) d\boldsymbol{\gamma} = \frac{1}{Q} \sum_{q=1}^Q c_k^q, \quad (22)$$

where c_k^q denotes the fit computed from the q th imputed data matrix (cf. Equation 12). As can be seen, the complexity computed from data with missing values is the average of the complexities computed for Q imputed data matrices.

Equations 17/18 and 21/22 can be used to compute the Bayes factor BF_{ku} displayed in Equation 10 from data with missing values. For the Bayes factors implemented in `BayesFactor` and those presented in Appendix A, the following representation will be used:

$$BF_{ku} = \frac{f_k}{c_k} = \frac{\frac{1}{Q} \sum_{q=1}^Q f_k^q}{\frac{1}{Q} \sum_{q=1}^Q c_k^q}. \quad (23)$$

As can be seen, the Bayes factor computed from data with missing values can be represented as the average of the fits computed from imputed data matrices divided by the average of the complexities computed from imputed data matrices. For the Bayes factor implemented in `Bain` the following representation will be used (cf. Equation 13):

$$BF_{ku} = \frac{f_k}{c_k} = \frac{\int_{\gamma \in H_k} \frac{1}{Q} \sum_{q=1}^Q g_u(\gamma \mid \mathbf{X}_m^q, \mathbf{X}_o) d\gamma}{\int_{\gamma \in H_k} \frac{1}{Q} \sum_{q=1}^Q h_u(\gamma + \gamma_{adj} \mid \mathbf{X}_m^q, \mathbf{X}_o) d\gamma} = \frac{\int_{\gamma \in H_k} g_u(\gamma \mid \mathbf{X}_o) d\gamma}{\int_{\gamma \in H_k} h_u(\gamma + \gamma_{adj} \mid \mathbf{X}_o) d\gamma}. \quad (24)$$

Imputation of the Missing Values

As was elaborated earlier in this paper, imputations of the missing values can be obtained using joint modeling of the variables in the imputation model as is, for example, implemented in the R packages `norm`, `cat`, and `mix`, or full conditional modeling as is, for example, implemented in the R packages `mice` and `mi`. In this paper all illustrations will use `norm` and/or `mice`.

In case of joint modeling imputations should be obtained by iteratively sampling from Equations 14 and 15. An approximation of these imputations can be achieved using a joint posterior distribution proportional to

$$p(\mathbf{X}_m, \mathbf{Z}_m, \mathbf{X}_o, \mathbf{Z}_o \mid \boldsymbol{\theta}, \boldsymbol{\eta}) h(\boldsymbol{\theta}, \boldsymbol{\eta}), \quad (25)$$

where $h(\boldsymbol{\theta}, \boldsymbol{\eta})$ is a standard uninformative prior distribution as is implemented in the R packages `norm`, `cat`, and `mix`. This posterior is asymptotically equal to Equation 14 which is proportional to

$$p(\mathbf{X}_m, \mathbf{Z}_m, \mathbf{X}_o, \mathbf{Z}_o \mid \boldsymbol{\theta}, \boldsymbol{\eta}) h_u(\gamma + \gamma_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_m, \mathbf{Z}_m, \mathbf{X}_o, \mathbf{Z}_o), \quad (26)$$

because for increasing sample size the effect of the different prior distributions disappears.

In case of full conditional modeling as is implemented in the R packages `mice` and `mi`, the following sampling scheme is used. Let \mathbf{x}^p denote one column from $[\mathbf{X}, \mathbf{Z}]$ for $p = 1, \dots, P + R$. It is required to specify a conditional imputation model $p(\mathbf{x}^p \mid \boldsymbol{\lambda}^p, \mathbf{x}^{-p})$ for

$p = 1, \dots, P + R$, in which $\boldsymbol{\lambda}^p$ denotes the parameters of the model used to impute \boldsymbol{x}^p . Note that \boldsymbol{x}^{-p} denotes all the columns in $[\boldsymbol{X}, \boldsymbol{Z}]$ except \boldsymbol{x}^p . Subsequently, the following procedure is used to create imputations:

1. Assign initial values, denoted by $q = 0$, to \boldsymbol{X}_m^q and \boldsymbol{Z}_m^q .
2. Iterate Steps 3 and 4 for $p = 1, \dots, P + R$ during $q = 1, \dots, Q$ iterations.
3. Sample $\boldsymbol{\lambda}^{p,q}$ from $g(\boldsymbol{\lambda}^p \mid \boldsymbol{x}^{p,q-1}, \boldsymbol{x}^{-p,q-1}) \propto p(\boldsymbol{x}^{p,q-1} \mid \boldsymbol{\lambda}^p, \boldsymbol{x}^{-p,q-1})h(\boldsymbol{\lambda}^p)$ where $h(\cdot)$ is a standard uninformative prior.
4. Sample $\boldsymbol{x}_m^{p,q}$ from $p(\boldsymbol{x}_m^p \mid \boldsymbol{\lambda}^{p,q}, \boldsymbol{x}^{1,q}, \dots, \boldsymbol{x}^{p-1,q}, \boldsymbol{x}^{p+1,q-1}, \dots, \boldsymbol{x}^{P+R,q-1})$.

Application of this procedure results in $q = 1, \dots, Q$ imputed data matrices $[\boldsymbol{X}_m^q, \boldsymbol{X}_o, \boldsymbol{Z}_m^q, \boldsymbol{Z}_o]$.

Example 1 Continued: Normal Linear Regression

When `norm` is used to impute the missing values the following procedure is used:

1. Assign initial values ($q = 0$) to $\boldsymbol{y}_m^q, \boldsymbol{x}_{1m}^q, \boldsymbol{x}_{2m}^q, \boldsymbol{z}_m^q$.
2. Iterate Steps 3 and 4 for $q = 1, \dots, Q$
3. Sample $\boldsymbol{\alpha}^q, \sigma^{2q}, \boldsymbol{\mu}_x^q, \boldsymbol{\Sigma}_x^q$ from a posterior distribution proportional to Equation 25 in which $p(\boldsymbol{X}_m, \boldsymbol{Z}_m, \boldsymbol{X}_o, \boldsymbol{Z}_o \mid \boldsymbol{\alpha}, \sigma^2, \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$ is equal to Equation 3 and $h(\boldsymbol{\alpha}, \sigma^2, \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$ is a standard uninformative prior.

4. Sample $\boldsymbol{y}_m^q, \boldsymbol{x}_{1m}^q, \boldsymbol{x}_{2m}^q, \boldsymbol{z}_m^q$ from $p(\boldsymbol{y}_m^q, \boldsymbol{x}_{1m}^q, \boldsymbol{x}_{2m}^q, \boldsymbol{z}_m^q \mid \boldsymbol{\alpha}, \sigma^2, \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$

the interested reader is referred to Shafer (1997, Chapters 5 and 6) for further elaborations.

When `mice` is used to impute the missing values the following procedure is used:

1. Assign initial values ($q = 0$) to $\boldsymbol{y}_m^q, \boldsymbol{x}_{1m}^q, \boldsymbol{x}_{2m}^q, \boldsymbol{z}_m^q$.
2. Iterate Steps 3 through 5 for $q = 1, \dots, Q$.
3. Sample $\boldsymbol{\lambda}^q$ from $g(\boldsymbol{\lambda}^q \mid \boldsymbol{y}^{q-1}, \boldsymbol{x}_1^{q-1}, \boldsymbol{x}_2^{q-1}, \boldsymbol{z}^{q-1}) \propto p(\boldsymbol{y}^{q-1} \mid \boldsymbol{x}_1^{q-1}, \boldsymbol{x}_2^{q-1}, \boldsymbol{z}^{q-1}, \boldsymbol{\lambda}^q)h(\boldsymbol{\lambda}^q)$,

where $p(\cdot)$ is based on

$$y_i^{q-1} = \lambda_0^q + \lambda_1^q x_{i1}^{q-1} + \lambda_2^q x_{i2}^{q-1} + \lambda_3^q z_i^{q-1} + e_i, \text{ with } e_i \sim \mathcal{N}(0, \lambda_4^q), \quad (27)$$

and $h(\boldsymbol{\lambda}^q) \propto 1/\lambda_4^q$.

4. Sample each missing value y_{im}^q from

$$p(y_{im} | x_{i1}^{q-1}, x_{i2}^{q-1}, z_i^{q-1}, \boldsymbol{\lambda}^q) \sim \mathcal{N}(\lambda_0^q + \lambda_1^q x_{i1}^{q-1} + \lambda_2^q x_{i2}^{q-1}, \lambda_4^q).$$

5. Repeat for $x_{i1,m}, x_{i2,m}, z_{im}$ steps analogous to Steps 3 and 4 based on

$$p(\mathbf{x}_1^{q-1} | \mathbf{y}^q, \mathbf{x}_2^{q-1}, \mathbf{z}^{q-1}, \boldsymbol{\lambda}_q), p(\mathbf{x}_2^{q-1} | \mathbf{y}^q, \mathbf{x}_1^q, \mathbf{z}^{q-1}, \boldsymbol{\lambda}_q), \text{ and } p(\mathbf{z}^{q-1} | \mathbf{y}^q, \mathbf{x}_1^q, \mathbf{x}_2^q, \boldsymbol{\lambda}_q),$$

respectively. The interested reader is referred to van Buuren (2012, Chapter 4) for further elaborations.

End Example 1 Continued

Computing ‘‘BayesFactor(s)’’ from Data with Missing Values

The Bayes factors implemented in the R package `BayesFactor` are mainly suited for the evaluation of hypotheses specified using equality constraints. Many of these Bayes factors can be computed using Equations 17 and 21 based on

$$g_u(\boldsymbol{\theta} | \mathbf{X}_m, \mathbf{X}_o) \propto p(\mathbf{X}_m, \mathbf{X}_o | \boldsymbol{\theta}) h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}). \quad (28)$$

Because $h_u(\cdot)$ is a user specified prior distribution that does not depend on the data and therefore does not vary over imputed data sets, Equation 23 can be rewritten as:

$$BF_{ku} = \frac{\frac{1}{Q} \sum_{q=1}^Q f_k^q}{\frac{1}{Q} \sum_{q=1}^Q c_k^q} = \frac{\frac{1}{Q} \sum_{q=1}^Q f_k^q}{c_k} = \frac{1}{Q} \sum_{q=1}^Q \frac{f_k^q}{c_k} = \frac{1}{Q} \sum_{q=1}^Q BF_{ku}^q, \quad (29)$$

where the fit computed from the q -th imputed data matrix

$$f_k^q = \int_{\boldsymbol{\theta} \in H_k} g_u(\boldsymbol{\theta} | \mathbf{X}_m^q, \mathbf{X}_o) d\boldsymbol{\theta} \quad (30)$$

and the complexity

$$c_k = \int_{\boldsymbol{\theta} \in H_k} h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}) d\boldsymbol{\theta}. \quad (31)$$

Stated otherwise, BF_{ku} can be computed from data with missing values as the average of the corresponding Bayes factors computed for each imputed data set. Note that, Equation 29 implies that it is not possible to compute BF_{uk} as the average of the corresponding

BF_{uk}^q s:

$$BF_{uk} = \frac{\frac{1}{Q} \sum_{q=1}^Q c_k^q}{\frac{1}{Q} \sum_{q=1}^Q f_k^q} = \frac{c_k}{\frac{1}{Q} \sum_{q=1}^Q f_k^q}, \quad (32)$$

which can not be rewritten in terms of BF_{uk}^q s. Therefore, to ensure transitivity relationships $BF_{uk} = 1/BF_{ku}$ and $BF_{kk'} = BF_{ku}/BF_{k'u}$.

Not all functions in the `BayesFactor` package use prior distributions that do not depend on the data. Examples for which this does hold are Bayes factors for t-tests and ANOVAs (Rouder et al. 2009; Rouder et al. 2012) but not Bayes factors for multiple regression (Rouder et al. 2012) because there the data for the predictors is used to scale the prior distribution. The approach presented in this paper can only be used with the `BayesFactor` package if the prior distribution is specified independently of the data.

Note that, as highlighted earlier, the approximate equality in Equation 29 is caused by the fact that the posterior distribution used by, for example, `norm` or `mice` to create multiple imputations is only asymptotically equal to Equation 28.

Computing the Approximate Adjusted Fractional Bayes Factor from Data with Missing Values

The Bayes factor implemented in `Bain` is suited for the evaluation of hypotheses of the general form displayed in Equation 4, that is, specified using equality and/or inequality constraints. It uses a fraction of the information in the likelihood to specify the prior distribution. When introducing the *unadjusted* fractional Bayes factor, O'Hagan (1995) used the following factorization:

$$p(\mathbf{X}_m, \mathbf{X}_o \mid \boldsymbol{\theta}) = c \times \ell(\mathbf{X}_m, \mathbf{X}_o \mid \boldsymbol{\theta})^{1-b(\mathbf{X}_m, \mathbf{X}_o)} \ell(\mathbf{X}_m, \mathbf{X}_o \mid \boldsymbol{\theta})^{b(\mathbf{X}_m, \mathbf{X}_o)}, \quad (33)$$

where c denotes the normalizing constant, and $\ell(\cdot)$ the likelihood function. The idea is to use a fraction $b(\mathbf{X}_m, \mathbf{X}_o)$ of the information in the likelihood function to implicitly specify a default prior distribution (Gilks, 1995). Usually the fraction $b(\mathbf{X}_m, \mathbf{X}_o)$ is chosen such that it corresponds to the size of a minimal training sample (Berger and Pericchi, 1996,

2004), that is, in the case of complete data, the smallest sample of size S out of N *complete* observations rendering a proper prior distribution: $b(\mathbf{X}_m, \mathbf{X}_o) = S/N$.

In line with Mulder (2014) this idea can be used to specify the adjusted fractional Bayes factor as a function of the prior and posterior distribution of $\boldsymbol{\theta}$ under H_u :

$$g_u(\boldsymbol{\theta} \mid \mathbf{X}_m, \mathbf{X}_o) \propto \ell(\mathbf{X}_m, \mathbf{X}_o \mid \boldsymbol{\theta})^{1-b(\mathbf{X}_m, \mathbf{X}_o)} h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega} \mid \mathbf{X}_m, \mathbf{X}_o), \quad (34)$$

where the adjusted fractional prior $h_u(\cdot \mid \cdot)$

$$h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega} \mid \mathbf{X}_m, \mathbf{X}_o) \propto \ell(\mathbf{X}_m, \mathbf{X}_o \mid \boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega})^{b(\mathbf{X}_m, \mathbf{X}_o)} h(\boldsymbol{\gamma}, \boldsymbol{\omega}) = g_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega} \mid \mathbf{X}_m, \mathbf{X}_o, b(\mathbf{X}_m, \mathbf{X}_o)), \quad (35)$$

is based on a standard uninformative prior $h(\cdot)$ and an adjusted mean for $\boldsymbol{\gamma}$. Note that, $g_u(\dots \mid \dots, b(\dots))$ denotes a prior distribution based on a fraction $b(\dots)$ of the information in the posterior distribution.

Bain (Gu et al., 2014; Gu, Mulder, and Hoijsink, in press) is an abbreviation of Bayesian inequality constrained hypotheses evaluation. It renders an approximation of the adjusted fractional Bayes factor based on a normal approximation of the posterior and prior distributions of the parameters $\boldsymbol{\gamma}$, that is, a normal approximation of Equation 34 and 35 with $\boldsymbol{\omega}$ integrated out. The posterior distribution of $\boldsymbol{\gamma}$ for each imputed data matrix $g_u(\boldsymbol{\gamma} \mid \mathbf{X}_m^q, \mathbf{X}_o)$ (see Equation 18) can be approximated by a multivariate normal distribution with maximum likelihood estimated mean $\hat{\boldsymbol{\gamma}}^q$ and corresponding covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\gamma}}^q$.

Standard results from the multiple imputation literature imply that the posterior distribution of $\boldsymbol{\gamma}$ is a summary of the estimates and covariance matrices obtained for the $q = 1, \dots, Q$ imputed data matrices (see, for example, Van Buuren, 2012, pp. 37-38):

$$g_u(\boldsymbol{\gamma} \mid \mathbf{X}_o) \approx \mathcal{N}(\boldsymbol{\gamma} \mid \bar{\boldsymbol{\gamma}}, \boldsymbol{\Sigma}_{\boldsymbol{\gamma}}), \quad (36)$$

where

$$\bar{\gamma} = \frac{1}{Q} \sum_{q=1}^Q \hat{\gamma}^q. \quad (37)$$

The covariance matrix Σ_{γ} is obtained using

$$\bar{\Sigma}_{\gamma} = \frac{1}{Q} \sum_{q=1}^Q \Sigma_{\gamma}^q \quad (38)$$

and

$$\mathbf{B} = \frac{1}{Q-1} \sum_{q=1}^Q (\gamma^q - \bar{\gamma})(\gamma^q - \bar{\gamma})^t \quad (39)$$

to obtain

$$\Sigma_{\gamma} = \bar{\Sigma}_{\gamma} + \left(1 + \frac{1}{Q}\right) \mathbf{B}. \quad (40)$$

Consequently, the fit f_k from Equation 18 can be approximated by

$$f_k = \int_{\gamma \in H_k} \mathcal{N}(\gamma \mid \bar{\gamma}, \Sigma_{\gamma}) d\gamma. \quad (41)$$

Equation 35 implies that the complexity c_k from equation 22 can be computed using

$$c_k = \int_{\gamma \in H_k} \mathcal{N}(\gamma \mid \gamma_B, \Sigma_{\gamma} / b(\mathbf{X}_o)) d\gamma, \quad (42)$$

Note that, due to the missing values the effective sample size of \mathbf{X}_o is no longer N but N_o .

The fraction of missing information λ can be used to compute $N_o = N - \lambda N$. To compute the fraction of missing information the following quantities are needed (see, for example,

Van Buuren, 2012, pp. 41-43):

$$\alpha = \left(1 + \frac{1}{Q}\right) \text{tr}(\mathbf{B}\Sigma^{-1})/w, \quad (43)$$

where w denotes the size of γ ,

$$\nu_{old} = \frac{Q-1}{\alpha^2}, \quad (44)$$

$$\nu_{com} = N - w, \quad (45)$$

$$\nu_{obs} = \frac{\nu_{com} + 1}{\nu_{com} + 3} \nu_{com} (1 - \alpha), \quad (46)$$

$$\nu = \frac{\nu_{old} \nu_{obs}}{\nu_{old} + \nu_{obs}}, \quad (47)$$

leading to

$$\lambda = \frac{\nu + 1}{\nu + 3}\alpha + \frac{2}{\nu + 3}. \quad (48)$$

For the Bayes factor implemented in `Bain`, $b(\mathbf{X}_o) = T/N_o$. Note that, T denotes the number of independent constraints imposed on γ in all K hypotheses under consideration, that is, the number of independent rows in $[\mathbf{S}_1, \dots, \mathbf{S}_K, \mathbf{R}_1, \dots, \mathbf{R}_K]$. To give one example: the number of independent constraints in $H_1 : \gamma_1 = \gamma_2 = \gamma_3$ and $H_2 : \gamma_1 > \gamma_2 > \gamma_3$ equals 2. The approximate adjusted fractional Bayes factor implemented in the package `Bain` is

$$BF_{ku} = f_k/c_k, \quad (49)$$

that is, Equation 24 with f_k and c_k defined in Equations 41 and 42, respectively. As elaborated before, BF_{ku} can be used to compute BF_{uk} and $BF_{kk'}$.

Example 1 Continued: Normal Linear Regression

This and the next three sections contain applications of the methodology developed in this paper. In this section Example 1 will be finished. The next section will show using a simple one variable example that the Bayes factor computed using the proposed methodology is indeed (approximately) equal to the Bayes factor that results if it is computed using only the observed values in \mathbf{X}_o and \mathbf{Z}_o (cf. Equation 13 and the subsequent derivation of f_k and c_k). The subsequent section will compute Bayes factor for a complete data matrix and compare it to the Bayes factor obtained using listwise deletion, the wrong imputation model, the correct imputation model and the correct imputation model extended with auxiliary variables that are not part of the missing data mechanism. The final application section will illustrate the versatility of the approach proposed by applying it to the evaluation of informative hypotheses in the context of a confirmatory factor model when the data contain missing values.

Example 1 will be finished using data from Stevens (1996, Appendix A) concerning the effect of the first year of the Sesame street series on the knowledge of 240 children in

the age range 34 to 69 months. We will use the following variables: \mathbf{y} , the knowledge of numbers after watching Sesame street; \mathbf{x}_1 , the knowledge of numbers before watching Sesame street; \mathbf{x}_2 , the knowledge of letters before watching Sesame street; and, \mathbf{z} , a test measuring the mental age of children.

The Sesame street data do not contain missing values. The following missing data mechanism was used to create missing values:

$$P(y_i = m) = \frac{1}{1 + \exp(-25 + x_{i1} + x_{i2})}; \quad (50)$$

$$P(x_{i1} = m, x_{i2} = m) = \frac{1}{1 + \exp(-32 + z_i)}; \quad (51)$$

and,

$$P(z_i = m) = .10 \text{ for } i = 1, \dots, N. \quad (52)$$

As can be seen, the probability that z_i is missing is .10. The probability that both x_{i1} and x_{i2} are missing increases if the score on z_i decreases. This reflects the idea that children with a lower mental age are less prone to attend the pretest. The probability that y_i is missing increases with decreasing x_{i1} and x_{i2} . This reflects the idea that children with a below average performance on the pre-test have a smaller probability of showing up during the post-test. Note that, this is a missing at random (MAR) missing data mechanism because the probability that a score on a variable is missing may depend on the scores on the other variables that are available, but *not* on the missing value itself.

In Table 1 descriptives of the resulting data matrix are displayed. As can be seen, about 25%, 15%, 15%, 10% of the observations are missing from \mathbf{y} , \mathbf{x}_1 , \mathbf{x}_2 , and \mathbf{z} , respectively. As can also be seen, the scales on which \mathbf{x}_1 and \mathbf{x}_2 are measured are comparable, which implies that it makes sense to compare their regression coefficients as is done in H_2 . Using `norm` and `mice` we created 1000 completed data matrices in which the missing values are imputed. At the end of this section the choice for 1000 imputations will be elaborated.

Summarizing the information in the 1000 completed data matrices imputed using `norm` rendered

$$\bar{\gamma} = [.644, .006], \quad (53)$$

$$\Sigma_{\gamma} = \begin{pmatrix} .011 & -.008 \\ -.008 & .014 \end{pmatrix}, \quad (54)$$

and the fraction of missing information $\lambda = .19$. Based on these quantities, `Bain` evaluated Equation 49 for H_1 and H_2 rendering $BF_{1u} = .00$ and $BF_{2u} = 7.94$, that is, there is no support in the data for H_1 and H_2 is 7.94 times as likely as H_u . As expected, the Bayes factors based on imputation using `mice` were virtually identical: $BF_{1u} = .00$ and $BF_{2u} = 7.97$. The data matrix used and annotated R code detailing the steps in the analyses presented in this section can be found in the on-line supplementary materials.

This relatively large number of $Q = 1000$ completed data matrices was chosen to ensure a stable estimate of the fraction of missing information λ . In Table 2, λ was computed using three different seeds to start the sampling of the missing values as executed by `mice`. As can be seen, for $Q = 10$ the λ may differ as much as .12. When Q is increased the differences become smaller. For $Q = 500$ the largest difference is about .02, which ensures that $Q = 1000$ renders accurate estimates of λ . In each example in this paper $Q = 1000$ will be used. Researchers applying the approach presented in this paper are well-advised to also use a large value for Q and to verify using different seeds for the multiple imputation package used that the value chosen renders a stable estimate of λ .

Example 2: Approximate Equality of Bayes Factors Computed from Observed Values and Completed Data Matrices

Consider the analysis model

$$x_i \sim \mathcal{N}(\gamma, \omega) \text{ for } i = 1, \dots, N, \quad (55)$$

and the informative hypotheses $H_1 : \gamma = 0$ and $H_2 : \gamma > 0$. Note that, there is only one γ and therefore that $H_u : \gamma$ is a hypothesis without restrictions on this gamma. Data will be

generated such that the sample average of the observed values in x is $\bar{x} \in \{-.2, 0, .2, .5\}$, and the sample standard deviation of the observed values is $s = 1$, the number of observed values is $N_o = 30$ and $N_m = 20$ missing values are added, that is, the missing values are MAR (in this case in fact missing completely at random). This setup allows for inference with (that is, based on $\mathbf{X}_o, \mathbf{X}_m$) and without (that is, based on \mathbf{X}_o) imputation of the missing values.

This setup enables the illustration of what was shown by means of equations (cf. Equation 13 and the subsequent derivation of f_k and c_k) earlier in this paper: if the missing values are MAR the Bayes factor computed using only the observed values and the Bayes factor computed from multiple completed data matrices are approximately identical. Note that, results are only approximately equal because the posterior distribution used for imputation is only asymptotically equal to the posterior distributions used to compute Bayes factors. Note furthermore, that the approximate equality was shown in general, but can only be illustrated using this simple setup because in more complicated setups like, for example, Example 1, the Bayes factor can be computed using the multiple imputation approach proposed in this paper, but there is no way in which the Bayes factor can directly be computed from the observed values like in the simple example presented in this section. Consider, for example, the regression model from Example 1 with missing values in \mathbf{y} , \mathbf{x}_1 , \mathbf{x}_2 , and the auxiliary variable \mathbf{z} needed to achieve MAR. Neither `BayesFactor` nor `Bain` can be used to compute Bayes factors using these data because they require the data to be complete. However, as elaborated earlier in this paper, this can be solved based on multiple imputation of the missing values.

Columns 3, 4, and 5 of Table 3 contain the input needed to use `Bain` for the evaluation of H_1 and H_2 (cf. Equations 41 and 42): $\bar{\gamma}$; Σ_γ ; and, λ . Each of these are estimated based on 1000 imputations of the missing values using `mice`. Note that, the imputation model was identical to the analysis model, and therefore the same irrespective of whether a joint or full conditional specification is used. As can be seen, the estimates

based on analysis of the observed values and multiple imputation are very similar (to two decimal places), but the fraction of missing information λ is slightly overestimated. The resulting values of the Bayes factors based on the observed values and multiple imputation (Equation 49) for BF_{1u} (column 5) and BF_{2u} (column 6) are similar. With a better estimate of λ they would have been virtually the same. Nevertheless, this simple example shows that the Bayes factor computed from the observed values can adequately be approximated by the imputed data based Bayes factor because the differences that can be observed do not change the interpretation of the resulting numbers.

Using the function `ttestBF` from the `BayesFactor` R package, the Bayes factors BF_{1u} and BF_{2u} (cf. Equation 29) are computed using only the observed values multiple imputation. In this package the analysis model is reparameterized as $x_i \sim \mathcal{N}(\sqrt{\omega}\gamma, \omega)$, where γ denotes the standardized effect size, $h_u(\gamma) \sim \text{Cauchy}(0, .707)$ where $.707$ is the prior scale, and $h(\omega) = 1/\omega$. As can be seen in the last two columns of Table 3, there is a close correspondence between the Bayes factors based on the observed values and multiple imputation.

In Figure 1 the BF_{2u}^q for $q = 1, \dots, 1000$ as computed using the `BayesFactor` R package are displayed for $\bar{x} = 0$, that is, one data matrix with missing values that was imputed 1000 times and for each imputed data matrix the Bayes factor was computed. As can be seen, using single imputation to obtain the Bayes factor is not a good idea because the BF_{2u}^q vary between 0 and 2.0. Using the BF_{2u}^q based on only one of these imputations would render a Bayes factor that is essentially a random number from the interval 0 to 2.0 which can easily be substantially different from the correct value 1.0.

Note that, the codes used to obtain the results in Table 3 and Figure 1 can be found in the on-line supplementary materials. Note furthermore that, the analyses presented in this section were also executed using N_o equal to 40 and 20 and N_m equal to 10 and 30, respectively. The results were completely in line with those described above.

Example 3: The Effect of the Choice of the Auxiliary Variables

This section will illustrate the effect of the choice of the auxiliary variables on the Bayes factor computed from data with missing values. One data matrix without missing values is used consisting of a dependent variable $\mathbf{y} = [y_1, \dots, y_i, \dots, y_N]$, three potential auxiliary variables $\mathbf{z}_1 = [z_{11}, \dots, z_{i1}, \dots, z_{N1}]$, $\mathbf{z}_2 = [z_{12}, \dots, z_{i2}, \dots, z_{N2}]$, and $\mathbf{z}_3 = [z_{13}, \dots, z_{i3}, \dots, z_{N3}]$, and a grouping variable $\mathbf{x} = [x_1, \dots, x_i, \dots, x_N]$, that attains the value 0 for the members of Group 1 and the value 1 for the members of Group 2. For both groups, the means and covariance matrix of the dependent and auxiliary variables are displayed in Table 4.

The analysis model of interest in this example is

$$y_i = \alpha_0 + \alpha_1 D_i + e_i \text{ with } e_i \sim \mathcal{N}(0, \sigma^2), \quad (56)$$

that is, a two group ANOVA in which α_1 denotes the difference in means between Groups 2 and 1 because D_i equals 0 if person i is a member of Group 1 and 1 if person i is a member of Group 2. The hypotheses of interest are $H_1 : \alpha_1 = 0$ and $H_2 : \alpha_1 > 0$. Missing data were created such that it holds in Group 1 that the smaller z_{i1} the larger the probability that y_i is missing and in Group 2 the larger z_{i1} the larger the probability that y_i is missing:

$$P(y_i = m \mid D_i = 0) = \frac{1}{1 + \exp(z_{i1})}, \quad (57)$$

and

$$P(y_i = m \mid D_i = 1) = \frac{1}{1 + \exp(-z_{i1})}. \quad (58)$$

This rendered 24 missing values in \mathbf{y} for both the members of Group 1 and Group 2.

The hypotheses H_1 and H_2 were evaluated using six scenarios:

1. Using the full data before the creation of missing values.
2. Using the data remaining after list-wise deletion of the cases with missing values.
3. Using the data after multiple imputation of the missing values using \mathbf{y} and \mathbf{x} without auxiliary variables in a multivariate normal imputation model (c.f. Equation 3).

Note that, \mathbf{x} can be modeled in this manner only because it does not contain missing values. Would it have contained missing values the imputation model should be one of the family of general location models (Shafer, 1997, Chapter 9).

4. Like Scenario 3 but now using \mathbf{y} , \mathbf{x} , and the variable predicting the missing values \mathbf{z}_1 .

5. Like Scenario 3 but now using \mathbf{y} , \mathbf{x} , \mathbf{z}_1 , and the variables not predicting the missing values \mathbf{z}_2 and \mathbf{z}_3 .

6. Like Scenario 3 but now using \mathbf{y} , \mathbf{x} , and only the variables not predicting the missing values \mathbf{z}_2 and \mathbf{z}_3 .

The results are displayed in Table 5. The following can be observed:

1. When the full data are used, the support in the data is substantially larger for H_2 than for H_1 ($\text{BF}_{21}=102.62$). This is not surprising, because $\hat{\alpha}_1 = .64$ and its standard deviation is $\sqrt{.033} = .18$. Note that, the results obtained in Scenarios 2 through 6 *cannot* straightforwardly be compared to the full data results because here the fraction of missing information equal 0, that is, $N_o = N = 100$, while in the other scenarios the fraction of missing information is substantially larger than 0. This will lead to a larger standard deviations for $\hat{\alpha}_1$ and smaller Bayes factors in the other scenarios. What can be learned from the full data is that scenarios that adequately account for the missing data mechanism should render estimates of $\hat{\alpha}$ close to .64, should render standard deviations larger than .18, and should render Bayes factors favoring H_2 .

2. Since the missing data mechanism (missingness depends on \mathbf{z}_1) is not accounted for in case of listwise deletion and imputation using only \mathbf{y} and \mathbf{x} the missing values are not MAR. This can for Scenarios 2 and 3 clearly be seen from $\hat{\alpha}_1 = .39$ (too small) and a BF_{21} slightly favoring H_1 instead of H_2 . This implies that inadequate representation of the missing data mechanism in the imputation model will render incorrect inferences.

3. As noted earlier, what is often done in practice is construction of an imputation model using the variables from the analysis model augmented with auxiliary variables that

are expected to be good predictors of the variables in the analysis model containing missing values. In Scenario 4 the imputation model uses \mathbf{y} , \mathbf{x} , and the variable \mathbf{z}_1 which explains the missing values. As can be seen, with the correct imputation model an accurate estimate $\hat{\alpha}_1 = .61$ is obtained and $\text{BF}_{21} = 3.18$ correctly indicates a preference for H_2 irrespective of the fact that 52% of the information in the data is missing leading to a standard deviation of $\sqrt{.077} = .28$ which is (and should be) substantially larger than .18 (the standard deviation obtained using the full data).

4. Scenario 5 is an extension of Scenario 4 in which also \mathbf{z}_2 and \mathbf{z}_3 are added to the variables in the imputation model. This is in line with Shafer and Graham (2002) who write "Although it is not necessary to have a scientific theory underlying an imputation model [which would imply that researchers *know* that \mathbf{z}_1 causes the missing values], it is crucial for that model to be general enough to preserve effects of interest in later analyses [that is, use the variables in the analysis model and variables related to the variables containing missing values]." As can be seen, even if "unnecessary" variables are included in the imputation model, an accurate estimate $\hat{\alpha}_1 = .65$ is obtained and $\text{BF}_{21} = 6.36$ correctly prefers H_2 over H_1 .

5. Continuing Scenario 5, even if \mathbf{z}_1 is not part of the imputation model but it does contain variables like \mathbf{z}_2 and \mathbf{z}_3 that are related to the variables containing missing values, an adequate estimate $\hat{\alpha}_1 = .60$ is obtained, and $\text{BF}_{21} = 4.64$ expresses a preference for H_2 . This is in line with "... that [imputation] model to be general enough to preserve effects of interest ..." (Shafer and Graham, 2002), that is, the imputation model does not necessarily have to contain the variables causing the missing values.

In summary, even if the missing data mechanism is unknown, using the variables in the analysis model extended with variables related to the variables containing missing values, may very well render adequate estimates of the effects of interest and Bayes factors adequately expressing preference for the hypotheses of interest. However, although researchers can (and should) argue in favor of their imputation model, MAR can never be

proven (Shafer and Graham, 2002) and all inferences are always conditional on the assumption of MAR. This does not only hold for the computation of Bayes factor from data with missing values, this holds for all inferences using data with missing values.

It is important to note that the Bayes factor is a quantification of information in a data set and not an estimate of a population quantity. Therefore, the uncertainty in the Bayes factor computed from data with missing values consists of two parts: (1) uncertainty caused by the fact that the missing data mechanism is unknown which has been elaborated in this section; and (2) Monte Carlo error. The latter can be controlled by choosing an appropriate value for Q as was elaborated at the end of the discussion of Example 1. Stated otherwise, there is no uncertainty due to sampling. Therefore, the variation in Bayes factors that can be observed in Figure 1 does not reflect uncertainty with respect to the value of the Bayes factor, it is a measure of the amount of information that is missing due to the presence of missing values: the larger the variation in Bayes factors computed for each imputed data matrix, the larger the amount of missing information.

Example 4: Confirmatory Factor Analysis

The Holzinger and Swineford (1939) dataset consists of mental ability scores of 301 seventh- and eighth-grade children. A subset with nine variables is widely used in the literature. It contains scores on: visual perception \mathbf{x}_1 ; cubes \mathbf{x}_2 ; lozenges \mathbf{x}_3 ; paragraph comprehension \mathbf{x}_4 ; sentence completion \mathbf{x}_5 ; word meaning \mathbf{x}_6 ; speeded addition \mathbf{x}_7 ; speeded counting of dots \mathbf{x}_8 ; and speeded discrimination of straight and curved capitals \mathbf{x}_9 . A common analysis model for these data is the following confirmatory factor model:

$$\begin{aligned}
 x_{pi} &= \alpha_p + \lambda_p t_{1i} + \epsilon_p \text{ for } p = 1, 2, 3, \\
 x_{pi} &= \alpha_p + \lambda_p t_{2i} + \epsilon_p \text{ for } p = 4, 5, 6, \\
 x_{pi} &= \alpha_p + \lambda_p t_{3i} + \epsilon_p \text{ for } p = 7, 8, 9,
 \end{aligned}
 \tag{59}$$

where: \mathbf{t}_1 denotes a visual factor, \mathbf{t}_2 a textual factor, and \mathbf{t}_3 a speed factor with

$$\begin{pmatrix} t_{1i} \\ t_{2i} \\ t_{3i} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & 1 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & 1 \end{pmatrix} \right). \quad (60)$$

Note that, the $\boldsymbol{\gamma} = [\gamma_{12}, \gamma_{13}, \gamma_{23}]$ contains the correlations between the factors and $\boldsymbol{\omega}$ contains the intercepts α_p , the factor loadings λ_p , and the residual variances from $\epsilon_p \sim \mathcal{N}(0, \sigma_p^2)$. The informative hypothesis of interest for this data set is

$$H_1 : \gamma_{12} > \gamma_{23}, \gamma_{13} > \gamma_{23}, \quad (61)$$

which expresses that the correlation between the textual and speed factor is the smallest of the three correlations. The Holzinger and Swineford (1939) data will be used to illustrate that the approach proposed in this paper can also be used for other models than multiple regression. Missing data were created by replacing each observation by a missing with a probability of .20. Descriptive statistics are presented in Table 6.

The imputation model used was analogous to the one used for the multiple regression example displayed in Equation 3, that is, a multivariate normal distribution for the nine mental test scores. In the context of structural equation modeling and thus also the confirmatory factor analysis model displayed in Equations 59 and 60 this implies that the missing values are imputed under a saturated model, that is, a model that will perfectly reproduce the observed covariance matrix of the nine mental test scores. As is well known from the structural equation modeling literature (see, for example, Bollen, 1989), such a model encompasses the analysis model, that is, the parameters of the analysis model are a function of the parameters of the imputation model. Furthermore, as is elaborated in Graham (2003) the analysis model can be extended into a ‘‘saturated correlations model’’ which is equivalent to the ‘‘saturated imputation model’’ to account for the presence of auxiliary variables. Stated otherwise, the parameters of the imputation model can be separated into the parameters of the analysis model plus extra parameters accounting for the presence of auxiliary variables, as is required for the approach presented in this paper.

Using `mice` to obtain 1000 imputed data matrices rendered $\hat{\gamma} = [.39, .47, .26]$.

Furthermore,

$$\Sigma\gamma = \begin{pmatrix} .0056 & -.0003 & .0012 \\ -.0003 & .0230 & .0061 \\ .0012 & .0061 & .0065 \end{pmatrix}, \quad (62)$$

and the fraction of missing information is .42, that is, the effective sample size $N_0 = 301 - .42 \times 301$. Note that, about 20% missing values and a fraction of missing information of .42 may seem out of line. However, in the case of the classical correlation γ_{xy} where the odd-numbered persons have a missing value for y and the even numbered persons have a missing value for x , there are 50% missing values, while the fraction of information missing with respect to γ_{xy} equals 1.0. Feeding this information into `Bain` rendered $BF_{1u} = 3.58$, that is, there is some support in the data for H_1 . If `norm` is used for imputation the resulting Bayes factor was almost identical, that is, $BF_{1u} = 3.58$. Note that the codes used to obtain the results presented in this section can be found in the on-line supplementary materials.

Discussion

The computation of Bayes factors used for hypotheses testing from data with missing values has not previously received attention in the literature. In this paper two specific Bayes factors have been presented. The adjusted fractional Bayes factor has been developed to evaluate informative hypotheses in the context of a wide variety of statistical models. It is implemented in the R package `Bain` (<http://informative-hypotheses.sites.uu.nl/software/>) which is currently being included in JASP (<https://jasp-stats.org/>). The R package `BayesFactor` (<http://bayesfactorpcl.r-forge.r-project.org/>) is a versatile tool for the evaluation of null-hypotheses and is also included in JASP. The results presented in this paper extend the applicability of these Bayes factors for hypothesis evaluation, because empirical data often contain missing values. As is elaborated in Appendix A, the derivations presented in

this paper can also be applied to other Bayes factors. However, since these are not implemented in software packages, or, the software package does not render the necessary information, these can not straightforwardly be used by psychological researchers.

It has to be noted that new options keep being added to R packages that can be used for multiple imputation. For example, recently, multiple imputation for two level models has been added to `mice`. Also both `Bain` and `BayesFactor` are packages that are actively maintained and to which new options keep being added. The interested reader is well-advised to monitor new developments in order to be up to date as to what is and isn't possible with these packages.

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Appendix A: Other Bayes Factors

Fractional Bayes Factors

The fractional Bayes factor (O’Hagan, 1995) is different from the adjusted fractional Bayes factor in that the location of the fractional prior is not adjusted. It can be computed using Equations 17 and 21 (the latter without adjustment of the prior mean, that is, $\gamma_{adj} = 0$) based on the following posterior and prior distributions:

$$g_u(\boldsymbol{\theta} \mid \mathbf{X}_m, \mathbf{X}_o) \propto \ell(\mathbf{X}_m, \mathbf{X}_o \mid \boldsymbol{\theta})^{1-b(\mathbf{X}_m, \mathbf{X}_o)} \ell(\mathbf{X}_m, \mathbf{X}_o \mid \boldsymbol{\theta})^{b(\mathbf{X}_m, \mathbf{X}_o)} h_u(\boldsymbol{\theta}), \quad (63)$$

and,

$$h_u(\boldsymbol{\theta} \mid \mathbf{X}_m, \mathbf{X}_o) \propto \ell(\mathbf{X}_m, \mathbf{X}_o \mid \boldsymbol{\theta})^{b(\mathbf{X}_m, \mathbf{X}_o)} h_u(\boldsymbol{\theta}), \quad (64)$$

with $h_u(\boldsymbol{\theta})$ a standard non-informative prior:

$$BF_{ku} \approx \frac{\frac{1}{Q} \sum_{q=1}^Q \int_{\boldsymbol{\theta} \in H_k} g_u(\boldsymbol{\theta} \mid \mathbf{X}_m^q, \mathbf{X}_o) d\boldsymbol{\theta}}{\frac{1}{Q} \sum_{q=1}^Q \int_{\boldsymbol{\theta} \in H_k} h_u(\boldsymbol{\theta} \mid \mathbf{X}_m^q, \mathbf{X}_o) d\boldsymbol{\theta}} = \frac{\frac{1}{Q} \sum_{q=1}^Q f_k^q}{\frac{1}{Q} \sum_{q=1}^Q c_k^q}, \quad (65)$$

that is, it is the ratio of the average of the fits f_k^q and the average of the complexities c_k^q computed with respect to Q imputed data matrices. Because the prior mean is not

adjusted, the fractional Bayes factor is only suited for the evaluation of informative hypotheses specified using equality constraints (Mulder, 2014). Since it is not implemented in a software package, it is not readily available.

Bayes Factors Based on Expected Posterior Priors

Perez and Berger (2002) discuss Bayes factors based on expected posterior priors. They can be computed using Equations 17 and 21 (the latter without adjustment of the prior mean, that is, $\gamma_{adj} = 0$) based on the following posterior and prior distributions::

$$g_u(\boldsymbol{\theta} \mid \mathbf{X}_m, \mathbf{X}_o) \propto \ell(\mathbf{X}_m, \mathbf{X}_o \mid \boldsymbol{\theta})h_u(\boldsymbol{\theta} \mid mts(\mathbf{X}_m, \mathbf{X}_o)), \quad (66)$$

where $h_u(\boldsymbol{\theta} \mid mts(\mathbf{X}_m, \mathbf{X}_o))$ denotes the expected posterior prior which is based on multiple minimal training samples (mts , Berger and Pericchi, 1996; 2004) obtained from $\mathbf{X}_m, \mathbf{X}_o$. In close analogy with the fractional Bayes factor where a fraction of the likelihood is used to specify the prior distribution, it follows that

$$BF_{ku} \approx \frac{\frac{1}{Q} \sum_{q=1}^Q \int_{\boldsymbol{\theta} \in H_k} g_u(\boldsymbol{\theta} \mid \mathbf{X}_m^q, \mathbf{X}_o) d\boldsymbol{\theta}}{\frac{1}{Q} \sum_{q=1}^Q \int_{\boldsymbol{\theta} \in H_k} h_u(\boldsymbol{\theta} \mid mts(\mathbf{X}_m^q, \mathbf{X}_o)) d\boldsymbol{\theta}} = \frac{\frac{1}{Q} \sum_{q=1}^Q f_k^q}{\frac{1}{Q} \sum_{q=1}^Q c_k^q}, \quad (67)$$

that is, the ratio of the average of the fits f_k^q and the average of the complexities c_k^q computed with respect to Q imputed data matrices. Mulder, Hoijsink, and Klugkist (2010) show that this Bayes factor is only suited for the evaluation of informative hypotheses specified using equality constraints. Since it is not implemented in a software package, it is not readily available.

Mulder, Hoijsink, and de Leeuw (2012) elaborate how the expected posterior prior can be adjusted such that it is suited for the evaluation of informative hypotheses specified using equality and inequality constraints. This prior is called the approximate adjusted expected posterior prior $h_u(w(\boldsymbol{\theta}) \mid mts(\mathbf{X}_m, \mathbf{X}_o))$, where $w(\cdot)$ denotes a transformation of $\boldsymbol{\theta}$ such that the prior distribution of $\boldsymbol{\theta}$ is normally distributed with mean $\boldsymbol{\theta}_B$ and restricted covariance matrix $\boldsymbol{\tau}_B$. This results in:

$$BF_{ku} \approx \frac{\int_{\boldsymbol{\theta} \in H_m} \frac{1}{Q} \sum_{q=1}^Q g_u(\boldsymbol{\theta} \mid \mathbf{X}_m^q, \mathbf{X}_o) d\boldsymbol{\theta}}{\int_{\boldsymbol{\theta} \in H_m} \frac{1}{Q} \sum_{q=1}^Q h_u(w(\boldsymbol{\theta}) \mid mts(\mathbf{X}_m^q, \mathbf{X}_o)) d\boldsymbol{\theta}} = \frac{\frac{1}{Q} \sum_{q=1}^Q f_k^q(w(\boldsymbol{\theta}))}{\frac{1}{Q} \sum_{q=1}^Q c_k^q(w(\boldsymbol{\theta}))}, \quad (68)$$

where the notation stresses that the fit $f_k^q(w(\boldsymbol{\theta}))$ and the complexity $c_k^q(w(\boldsymbol{\theta}))$ are computed using the q -th imputed data matrix and a $w(\boldsymbol{\theta})$ that is based on the information in \mathbf{X}_o and therefore the same for each imputed data matrix.

The approximate adjusted expected posterior prior is implemented in the software package BIEMS (<http://informative-hypotheses.sites.uu.nl/software/>) As is shown in Equation 68, what is needed to compute the corresponding Bayes factor from data with missing values are the fit and complexity based on a $w(\boldsymbol{\theta})$ that is the same for each of the $q = 1, \dots, Q$ imputed data matrices. BIEMS only renders the fit and complexity (and only for inequality constrained hypotheses) based on $w^q(\boldsymbol{\theta})$, that is, the transformation is determined using the q -th imputed data matrix and not using only the observed values. Therefore, BIEMS can not be used to compute Equation 68. Researchers currently using BIEMS for the evaluation of informative hypotheses are advised to change to `Bain` or `BayesFactor` if their data are incomplete.

Appendix B: Derivation of the Bayes Factor for Hypothesis Evaluation

Chib's (1995) representation of the Bayes factor comparing the informative hypothesis H_k with the unconstrained hypothesis H_u can be written as:

$$BF_{ku} = \frac{p(\mathbf{X}_o, \mathbf{Z}_o \mid \boldsymbol{\theta}, \boldsymbol{\eta}) h_k(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o)}{g_k(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o)} / \frac{p(\mathbf{X}_o, \mathbf{Z}_o \mid \boldsymbol{\theta}, \boldsymbol{\eta}) h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o)}{g_u(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o)}, \quad (69)$$

where, $h_k(\cdot)$ and $h_u(\cdot)$ denote the prior distribution of the parameters for the informative and unconstrained hypotheses, respectively, and $g_k(\cdot)$ and $g_u(\cdot)$ denote the posterior distribution for the informative and unconstrained hypotheses, respectively.

As is highlighted in Equation 4, H_k for $k = 1, \dots, K$ are hypotheses obtained by imposing equality and/or inequality constraints on a parameter vector $\boldsymbol{\gamma} = [\gamma_1, \dots, \gamma_J]$. Each of these hypotheses is nested with $H_u : \gamma_1, \dots, \gamma_J$, that is, an hypothesis without constraints on the parameters.

The prior distributions $h_k(\cdot)$ and $g_k(\cdot)$ are those parts of $h_u(\cdot)$ and $g_u(\cdot)$, respectively, in agreement with H_k (Hoijsink, 2012, p. 59), that is,

$$\begin{aligned}
 h_k(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) &= \\
 \frac{h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) I(\boldsymbol{\gamma} \in H_k)}{\int_{\boldsymbol{\gamma}, \boldsymbol{\omega}, \boldsymbol{\eta}} h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) I(\boldsymbol{\gamma} \in H_k) d\boldsymbol{\gamma} d\boldsymbol{\omega} d\boldsymbol{\eta}} &= \\
 \frac{h_u(\boldsymbol{\gamma} + \boldsymbol{\gamma}_{adj}, \boldsymbol{\omega}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) I(\boldsymbol{\gamma} \in H_k)}{c_k}, & \tag{70}
 \end{aligned}$$

where $I(\cdot)$ equals one if the argument is true and zero otherwise, and,

$$\begin{aligned}
 g_k(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) &= \frac{g_u(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) I(\boldsymbol{\theta} \in H_k)}{\int_{\boldsymbol{\theta}, \boldsymbol{\eta}} g_u(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) I(\boldsymbol{\theta} \in H_k) d\boldsymbol{\theta} d\boldsymbol{\eta}} = \\
 \frac{g_u(\boldsymbol{\theta}, \boldsymbol{\eta} \mid \mathbf{X}_o, \mathbf{Z}_o) I(\boldsymbol{\theta} \in H_k)}{f_k}. & \tag{71}
 \end{aligned}$$

Substitution of $h_k(\cdot)$ and $g_k(\cdot)$ in Equation 69 with Equation 70 and 71, respectively, renders

$$BF_{ku} = f_k/c_k. \tag{72}$$

Table 1

Descriptive statistics for the Sesame Street Data with Missing Values

Variable	Mean	Minimum	Maximum	Number Missing
y	33.84	7	54	57
x_1	22.52	2	52	37
x_2	16.58	1	55	37
z	46.77	27	99	27

Table 2

The Choice of Q and the Stability of λ

Q	Seed = 999	Seed = 1234	Seed = 4321
10	.402	.276	.285
50	.272	.291	.299
100	.309	.278	.312
500	.300	.289	.305
1000	.308	.298	.293

Table 3

*Bayes Factors Computed Using only the Observed Values and Multiple Imputation with the Packages **Bain** and **BayesFactor***

	\bar{x}	$\bar{\gamma}$	Σ_{γ}	λ	Bain		BayesFactor	
					BF_{1u}	BF_{2u}	BF_{1u}	BF_{2u}
observed	-.2	-0.20	.03		3.01	.27	2.98	.30
imputed	-.2	-0.20	.03	.45	3.03	.29	2.94	.30
observed	0	.00	.03		5.47	1	5.14	1
imputed	0	.00	.03	.45	5.26	1	5.09	1.02
observed	.2	.20	.03		3.01	1.72	2.98	1.70
imputed	.2	.20	.03	.45	2.87	1.73	2.78	1.72
observed	.5	.50	.03		.13	1.99	.23	1.99
imputed	.5	.50	.03	.45	.13	1.99	.19	1.99

Table 4

Per Group, Means and Covariance Matrix of the Dependent and Auxiliary Variables

Variable	Group 1					Group 2				
	Mean	Covariance Matrix				Mean	Covariance Matrix			
\mathbf{y}	-.20	.72				.44	.91			
\mathbf{z}_1	-.03	.18	.74			-.29	.32	.98		
\mathbf{z}_2	-.14	.40	.36	1.16		.02	.60	.51	1.25	
\mathbf{z}_3	-.15	.32	.32	.51	.80	-.02	.57	.50	.86	1.36

Table 5

Key Quantities and Bayes Factors for Six Scenarios

Scenario	$\hat{\alpha}_1$	Σ_{α_1}	λ	BF_{1u}	BF_{2u}	BF_{21}
1	.64	.033	0	.02	2	102.62
2	.39	.070	.43	2.39	1.86	.78
3	.39	.074	.51	2.48	1.85	.75
4	.61	.077	.52	.62	1.97	3.18
5	.65	.067	.45	.31	1.98	6.36
6	.60	.063	.43	.43	1.98	4.64

Table 6

Descriptive statistics for the Holzinger and Swineford (1939) Data

Variable	Mean	Minimum	Maximum	Number Missing
\mathbf{x}_1	4.88	.67	8.5	53
\mathbf{x}_2	6.07	2.25	9.25	59
\mathbf{x}_3	2.30	.25	4.50	72
\mathbf{x}_4	3.07	.00	6.33	43
\mathbf{x}_5	4.35	1.00	6.50	74
\mathbf{x}_6	2.16	.14	6.14	62
\mathbf{x}_7	4.23	1.87	7.44	57
\mathbf{x}_8	5.48	3.50	8.30	58
\mathbf{x}_9	5.32	2.78	9.25	63

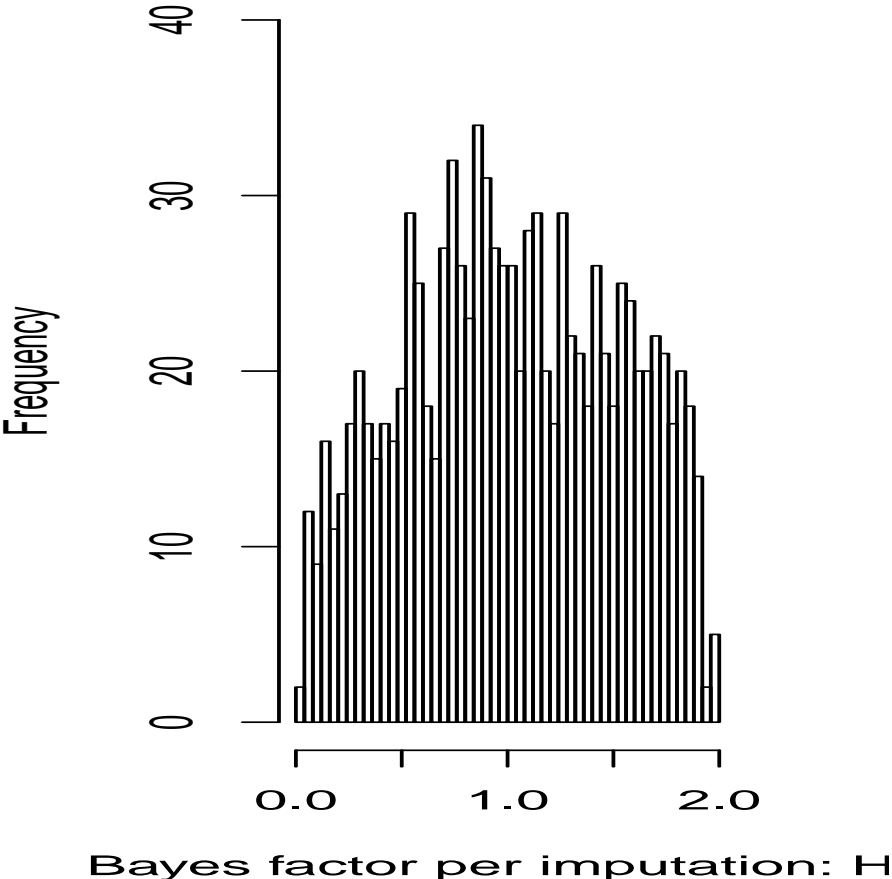


Figure 1. The distribution of BF_{2u}^q for $q = 1, \dots, 1000$ Imputations of One Data Set with Missing Values for which $\bar{x} = 0$.